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Robust POD Model Stabilization for the 3D Boussinesq Equations Based on Lyapunov Theory and Extremum Seeking

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***Abstract*— We present new results on robust model reduction for partial differential equations. Our contribution is threefold: 1.) The stabilization is achieved via closure models for reduced order models (ROMs), where we use Lyapunov robust control theory to design a new stabilizing closure model that is robust with respect to parametric uncertainties; 2.) The free parameters in the proposed ROM stabilization method are auto-tuned using a data-driven multi-parametric extremum seeking (MES) optimization algorithm; and 3.) The challenging 3D Boussinesq equation numerical test-bed is used to demonstrate the advantages of the proposed method.**

I. INTRODUCTION

A well known problem in model reduction for partial differential equations (PDEs) is the so-called *stable model reduction* problem. The goal is to use Galerkin projection onto a suitable set of modes to reduce PDEs to a small system of ordinary differential equations (ODEs), while maintaining the main characteristics of the original model, such as stability and prediction precision.

In this paper, we focus on reduced order models obtained by the method of proper orthogonal decomposition (POD) [1], which has been widely used to obtain surrogate models of tractable size in fluid flow applications. However, it has been observed that POD reduced order models can loose stability even when the flow solutions are stable. This is due to truncation of the stabilizing modes in the system [2]–[6].

We address the stable model reduction problem by using closure models, which are classically in the form of additive viscosity-like terms introduced in the reduced order models (ROMs) to ensure stability, by simple damping effect. Contrary to the existing closure models, which are commonly obtained from physics-based intuitions, we propose here a constructive approach, through means of robust Lyapunov theory, to design a new closure model that is robust to parametric uncertainties in the model. Furthermore, the obtained closure model has free parameters, which we auto-tune with a data-driven extremum seeking algorithm to optimally match predictions of the PDE model. The idea of using extremum-seeking to auto-tune closure models has been introduced by the authors in [7], on classical viscosity-based closure models. However, the difference with this work lies in the introduction of new robust closure models, based on robust control theory. In [7], we considered the one dimensional Burgers' equation. Here, we study the challenging case of 3D Boussinesq equations, which is a good model for a number of important control applications, e.g., airflow in HVAC systems [8]. However, it is also well known in the fluid dynamics community that 3D Boussinesq equations are hard to simulate, let alone, reduce and stabilize.

Our work extends existing results in the field. Indeed, stable model reduction of Navier-Stokes flow models by adding a nonlinear viscosity term to the reduced order model is considered in [9]. In [10], [11], incompressible flows are stabilized by an iterative search of the projection modes that satisfy a local Lyapunov stability condition. An optimization based approach for the POD modes of linear models, which solely focused on matching the outputs of the models is derived in [4], [6]. Kalb and Deane [3] added error correction terms to the reduced order model for improved accuracy and stabilization. Moreover, the authors in [2] calibrated the POD model by solving a quadratic optimization problem based on three different weighted error norms. Stable model reduction using closure models were proposed for the Boussinesq equations in [12]–[14]. These closure models modify some stability-enhancing coefficients of the reduced order ODE model using either constant additive terms, such as the constant eddy viscosity model, or time and space varying terms, such as Smagorinsky models (cf. [15]). The amplitudes of the additional terms are tuned in such a way to accurately stabilize the reduced order model.

However, such closure models, classically motivated from physics, *do not take into account parametric uncertainties in the model*, and their tuning is not always straightforward. Our work addresses these issues and proposes, based on robust nonlinear control theory, a new closure model in Section III, which includes parametric uncertainties in its formulation. Furthermore, we achieve (online) optimal auto-tuning of this closure model using a learning-based extremum seeking approach. Finally, we demonstrate the performance of proposed approach using the challenging 3D Boussinesq equation in Section IV. The paper ends with conclusions and future steps discussed in Section V. First, let us introduce some notations and necessary definitions.

II. BASIC NOTATIONS AND DEFINITIONS

For a vector $q \in \mathbb{R}^n$, the transpose is denoted by q^*. The Euclidean vector norm for $q \in \mathbb{R}^n$ is denoted by $\|\cdot\|$ so that $\|q\| = \sqrt{q^*q}$. The maximum eigenvalue of a matrix M is denoted by $\lambda_{max}(M)$. The Frobenius norm of a tensor $A \in \mathbb{R}^{\otimes_i n_i}$, with elements $a_{\mathbf{i}} = a_{i_1 \cdots i_k}$, is defined as $\|A\|_F \triangleq \sqrt{\sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} |a_{\mathbf{i}}|^2}$. The Kronecker delta function is

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defined as: $\delta_{ij} = 0$, for $i \neq j$ and $\delta_{ii} = 1$. We call a function analytic in a given set, if it admits a convergent Taylor series approximation in some neighborhood of every point of the set. Our PDEs (the Boussinesq equations) are solved on the unit cube $x \in \Omega = (0,1)^3$ and $t \in (0, t_f)$. We shall abbreviate the time derivative by $\dot{f}(t,x) = \frac{\partial}{\partial t} f(t,x)$, and consider the following Hilbert spaces: $\mathcal{H} = L^2(\Omega)$, $\mathcal{V} = H^1_{\text{div}}(\Omega) \subset (\mathcal{H})^3$ for velocity and $\mathcal{T} = H^1(\Omega) \subset \mathcal{H}$ for temperature. Thus, $\mathcal{V}$ is the space of divergence-free vector fields on Ω with components in $H^1(\Omega)$. Dirichlet boundary conditions are also considered in $\mathcal{V}$ and $\mathcal{T}$. We define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the associated norm $\| \cdot \|_{\mathcal{H}}$ on $\mathcal{H}$ as $\|f\|^2_{\mathcal{H}} = \int_\Omega |f(x)|^2 dx$, and $\langle f, g \rangle_{\mathcal{H}} = \int_\Omega f(x) g(x) dx$, for $f, g \in \mathcal{H}$. A function $T(t,x)$ is in $L^2([0,t_f]; \mathcal{H})$ if for each $0 \leq t \leq t_f$, $T(t, \cdot) \in \mathcal{H}$, and $\int_0^{t_f} \|T(t,\cdot)\|^2_{\mathcal{H}} dt < \infty$ with analogous definitions for the vector valued functions in $(\mathcal{H})^3$. To generalize the discussion below, we consider the abstract Hilbert space $\mathcal{Z}$, and later specialize to $\mathcal{Z} = \mathcal{V} \oplus \mathcal{T}$ when considering the Boussinesq equations. Finally, in the remainder of this paper we consider the stability of dynamical systems in the sense of Lagrange, e.g., [16]: A system $\dot{q} = f(t,q)$ is said to be Lagrange stable if for every initial condition q_0 associated with the time instant t_0, there exists $\epsilon(q_0)$, such that $\|q(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$.

III. Lyapunov-based robust stable model reduction of PDEs

A. Reduced Order PDE Approximation

We consider a stable dynamical system modeled by a nonlinear partial differential equation of the form

$$\dot{z} = \mathcal{F}(z), \qquad z(0) \in \mathcal{Z}, \tag{1}$$

where $\mathcal{Z}$ is an infinite-dimensional Hilbert space. Solutions to (1) can be approximated in an n-dimensional subspace $\mathcal{Z}^n \subseteq \mathcal{Z}$ through expensive numerical discretization. The approximate solution is denoted by $z_n(t,x) \in \mathcal{Z}^n$, where $t \in \mathbb{R}_+$ is the time variable, and $x \in \Omega$ denotes the space variable. In this paper, we use the well-established finite element method (FEM) for spatial discretization of (1), and refer the reader to [17] for details.

In many systems, particularly fluid flows considered herein, solutions of (1) evolve in a much lower-dimensional subspace $\mathcal{Z}^r \subseteq \mathcal{Z}^n$, spanned by only a few suitable (optimal) basis [1] functions. This gives rise to reduced order modeling through Galerkin projection, which follows three main steps, which we briefly outline below: First, one (spatially) discretizes the PDE using a finite, but large, number of basis functions. Second, one determines a set of r (often $r < 100$) spatial basis vectors $\phi_i(x) \in \mathcal{Z}^n$, that approximate the discretized PDE solution as $z_n(t,x)$ with respect to a pre-specified accuracy criterion. Third, a Galerkin projection onto the subspace spanned by the ϕ_i yields an r-dimensional ordinary differential equation for the time-dependent coefficient functions of the ROM expansion.

B. Proper Orthogonal Decomposition for ROMs

We briefly review the necessary steps for computing POD reduced order models, described in detail in [1], [18]. POD-based models are known for retaining a maximal amount of energy in the reduced model. The POD basis is computed from a collection of s time snapshots of the spatially discretized dynamical system:

$$\mathcal{S} = \{z_n(t_1,\cdot), ..., z_n(t_s,\cdot)\} \subset \mathcal{Z}^n. \tag{2}$$

The t_i are time instances at which snapshots are recorded, which do not have to be uniform. The *correlation matrix* K is then defined as

$$K_{ij} = \frac{1}{s} \langle z_n(t_i,\cdot), z_n(t_j,\cdot) \rangle_{\mathcal{H}}, \ i,j = 1, ..., s. \tag{3}$$

The normalized eigenvalues and eigenvectors of K are denoted by λ_i and v_i, respectively. Note that the λ_i are also referred to as the *POD eigenvalues*. The ith *POD basis function* is given by

$$\phi_i(x) = \frac{1}{\sqrt{s}\sqrt{\lambda_i}} \sum_{j=1}^{s} v_{i,j} z_n(t_j, x), \ i = 1, ..., r, \tag{4}$$

where $r \leq \min\{s, n\}$, the number of retained POD basis functions, depends on the application. The POD basis functions are orthonormal:

$$\langle \phi_i, \phi_j \rangle_{\mathcal{H}} = \int_\Omega \phi_i(x)^* \phi_j(x) dx = \delta_{ij}, \tag{5}$$

where δ_{ij} denotes the Kronecker delta function. In the POD basis, the solution of the PDE (1) can then be approximated by

$$z_n(t,x) = \sum_{i=1}^{r} q_i(t) \phi_i(x) \tag{6}$$

where $q_i(t), i = 1, \ldots, r$ are the time-dependent POD coefficients, whose dynamics are determined by Galerkin-projection of (1) onto the rth-order POD subspace $\mathcal{Z}^r \subseteq \mathcal{Z}^n \subseteq \mathcal{Z}$. This leads to an ODE of the form

$$\dot{q}(t) = F(q(t)) \in \mathbb{R}^r. \tag{7}$$

A projection of the initial condition for $z(0)$ can be used to determine $q(0)$.

C. Closure models for ROM stabilization

Consider a parameter-dependent PDE of the form

$$\dot{z}(t) = \mathcal{F}(z(t), \mu), \quad z(0) = z_0 \in \mathcal{Z}, \qquad \mu \in \mathbb{R}, \tag{8}$$

where parameter μ is assumed to be critical for the stability and accuracy of the model; changing the parameter can either make the model unstable, or inaccurate for prediction. As an example, since we are interested in fluid dynamics problems, μ can represent a viscosity coefficient. We assume that (8) satisfies the following

Assumption 1: The solutions of (8) are in $L^2([0,\infty); \mathcal{Z})$, for all $\mu \in \mathbb{R}$.

The corresponding reduced-order, parameter-dependent model takes the form:

$$\dot{q}(t) = F(q(t), \mu). \tag{9}$$

It is well known that Galerkin POD ROM (denoted POD ROM-G) can produce solutions that become unbounded at a finite time, despite the fact that the solution of (8) is bounded.

One of the main ideas behind closure modeling is that the viscosity coefficient μ in (9) can be replaced by a virtual viscosity coefficient μ_{cl}, whose form is chosen to stabilize the solutions of (9). Furthermore, a penalty term $H(\cdot,\cdot)$ is added to the original POD ROM-G, as follows

$$\dot{q}(t) = F(q(t), \mu_{cl}) + H(t, q(t)). \tag{10}$$

The choice of $H(\cdot,\cdot)$ is often motivated by physics, and depends on the structure of $F(\cdot,\cdot)$. For instance, one can use the Cazemier penalty model described in [20]. However, as we mentioned earlier, the available choices of H are not developed via a constructive method and never take into account parametric uncertainties. We propose to improve this in the next section.

D. Main result 1: Lyapunov-based Closure Model

We introduce here the first main result of this paper, namely a Lyapunov-based closure model which is robust to parametric uncertainties. To do so, we isolate the linear viscous term of the ROM (9) as follows

$$\dot{q}(t) = F(q(t), \mu) = \tilde{F}(q(t)) + \mu\, Dq(t), \tag{11}$$

where $D \in \mathbb{R}^{r\times r}$ represents a constant viscosity damping matrix, and the function $\tilde{F}(\cdot)$ represents the remainder of the ROM, i.e., the part without damping.

We consider here the case where $\tilde{F}$ might be unknown but bounded by a known function. This includes the case of parametric uncertainties in (8), which lead to structured uncertainties in (11). In this case, we use Lyapunov theory to propose a nonlinear closure model that (robustly) stabilizes the ROM in the sense of Lagrange stability.

We assume that $\tilde{F}$ satisfies the following assumption.

Assumption 2: The norm of the vector field $\tilde{F}$ is bounded by a known function of q, i.e., $\|\tilde{F}(q)\| \leq \tilde{f}(q)$.
We now present the following result.

Theorem 1: Consider the PDE (8), under Assumption 1, together with its stabilized ROM

$$\dot{q}(t) = \widetilde{F}(q(t)) + \mu_{cl}\, Dq(t) + H(q(t)), \tag{12}$$

where $\tilde{F}$ satisfies Assumption 2, the diagonal elements of D are negative, and μ_{cl} is given by

$$\mu_{cl} = \mu + \mu_e, \tag{13}$$

where μ is the nominal value of the viscosity coefficient in (8), and μ_e is the additional constant term. Then, the nonlinear closure model

$$H(q) = \mu_{nl}\widetilde{f}(q)\ \mathrm{diag}(d_{11}, ..., d_{rr})\ q, \quad \mu_{nl} > 0 \tag{14}$$

stabilizes the solutions of the ROM to the invariant set

$$\mathcal{S} = \{q \in \mathbb{R}^r\ s.t.\ \mu_{cl}\frac{\lambda_{\max}(D)\|q\|}{\widetilde{f}(q)} + \mu_{nl}\|q\| \max\{d_{11}, \ldots, d_{rr}\} + 1 \geq 0\}.$$

Proof 1: The proof has been removed due to space constraints, but will appear in a longer version of this work.

E. Main result 2: Multi-parametric extremum seeking (MES)-based closure model auto-tuning

As mentioned in our introduction and in [14], the tuning of the closure model amplitude is important to achieve an optimal stabilization of the ROM. To achieve optimal stabilization, we use model-free MES optimization algorithms to tune the coefficients of the closure models presented in Section III-C. The advantage of using MES is the auto-tuning capability that such algorithms allow. Moreover, in contrast to manual offline tuning approaches, the use of MES allows us to constantly tune the closure model, even during online operation of the system. Indeed, MES can be used offline to tune the closure model, but it can also be connected online to the real system to continuously fine-tune the closure model coefficients, such as the amplitudes of the closure models. Thus, the closure model can be valid for a longer time interval compared to the classical closure models with constant coefficients, which are usually tuned offline over a fixed finite time interval.

We start by defining a suitable learning cost function. The goal of the learning (or tuning) is to enforce Lagrange stability of the ROM (9), and to ensure that the solutions of the ROM (9) are close to those of the original PDE (8). The latter learning goal is important for the accuracy of the solution. Model reduction works toward obtaining a simplified ODE model which reproduces the solutions of the original PDE (the real system) with much less computational burden, i.e., using the lowest possible number of modes. However, for model reduction to be useful, the solution should be accurate.

We define the learning cost as a positive definite function of the norm of the error between the approximate solutions of (8) and the ROM (10),

$$\begin{aligned} Q(\widehat{\boldsymbol{\mu}}) &= \widetilde{H}(e_z(t, \widehat{\boldsymbol{\mu}})), \\ e_z(t, \widehat{\boldsymbol{\mu}}) &= z_n^{pod}(t, x; \widehat{\boldsymbol{\mu}}) - z_n(t, x; \boldsymbol{\mu}), \end{aligned} \tag{15}$$

where $\widehat{\boldsymbol{\mu}} = [\widehat{\mu}_e, \widehat{\mu}_{nl}]^* \in \mathbb{R}^2$ denotes the learned parameters, and $\widetilde{H}(\cdot)$ is a positive definite function of e_z. Note that the error e_z could be computed offline using solutions of the ROM (12), and exact (e.g., FEM-based) solutions of the PDE (8). The error could be also computed online where the $z_n^{pod}(t, x; \widehat{\boldsymbol{\mu}})$ is obtained from solving the ROM (12) online, and the $z_n(t, x; \boldsymbol{\mu})$ are real measurements of the system at selected spatial locations $\{x_i\}$. The latter approach would circumvent the FEM model, and directly operate on the system, making the reduced order model more consistent with respect to the realtime operating plant.

A practical way to implement the MES-based tuning of $\widehat{\boldsymbol{\mu}}$, is to begin with an offline tuning of the closure model.

One then uses the obtained ROM (with the optimal values of $\widehat{\boldsymbol{\mu}}$, namely $\boldsymbol{\mu}^{\mathrm{opt}}$) in the online operation of the system, e.g., control and estimation. We can then fine-tune the ROM online by continuously learning the best value of $\widehat{\boldsymbol{\mu}}$ at any given time during the operation of the system.

To derive formal convergence results, we introduce some classical assumptions on the learning cost function.

Assumption 3: The cost function $Q(\cdot)$ in (15) has a local minimum at $\widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}^{\mathrm{opt}}$.

Assumption 4: The cost function $Q(\cdot)$ in (15) is analytic and its variation with respect to $\boldsymbol{\mu}$ is bounded in the neighborhood of $\boldsymbol{\mu}^{\mathrm{opt}}$, i.e., $\|\nabla_{\boldsymbol{\mu}} Q(\widetilde{\boldsymbol{\mu}})\| \leq \xi_2,\ \xi_2 > 0$, for all $\widetilde{\boldsymbol{\mu}} \in \mathcal{N}(\boldsymbol{\mu}^{\mathrm{opt}})$, where $\mathcal{N}(\boldsymbol{\mu}^{\mathrm{opt}})$ denotes a compact neighborhood of $\boldsymbol{\mu}^{\mathrm{opt}}$.

Under these assumptions the following lemma holds.

Lemma 1: Consider the PDE (8) under Assumption 1, together with its ROM (12), (13), and (14). Furthermore, suppose the closure model amplitudes $\widehat{\boldsymbol{\mu}} = [\hat{\mu}_e, \hat{\mu}_{nl}]^*$ are tuned using the MES algorithm

$$\begin{aligned} \dot{y}_1(t) &= a_1 \sin\left(\omega_1 t + \frac{\pi}{2}\right) Q(\widehat{\boldsymbol{\mu}}), \\ \widehat{\mu}_e(t) &= y_1 + a_1 \sin\left(\omega_1 t - \frac{\pi}{2}\right), \\ \dot{y}_2(t) &= a_2 \sin\left(\omega_2 t + \frac{\pi}{2}\right) Q(\widehat{\boldsymbol{\mu}}), \\ \widehat{\mu}_{nl}(t) &= y_2 + a_2 \sin\left(\omega_2 t - \frac{\pi}{2}\right), \end{aligned} \tag{16}$$

where $\omega_{\max} = \max(\omega_1, \omega_2) > \omega^{\mathrm{opt}}$, ω^{opt} large enough, and $Q(\cdot)$ is given by (15). Let $e_{\boldsymbol{\mu}}(t) := [{\boldsymbol{\mu}_e}^{\mathrm{opt}} - \widehat{\boldsymbol{\mu}}_e(t), {\boldsymbol{\mu}_{nl}}^{\mathrm{opt}} - \widehat{\boldsymbol{\mu}}_{nl}(t)]^*$ be the error between the current tuned values, and the optimal values $\boldsymbol{\mu}_e^{\mathrm{opt}}$, $\boldsymbol{\mu}_{nl}^{\mathrm{opt}}$. Then, under Assumptions 3, and 4, the norm of the distance to the optimal values admits the following bound

$$\|e_{\boldsymbol{\mu}}(t)\| \leq \tfrac{\xi_1}{\omega_{\max}} + \sqrt{a_1^2 + a_2^2},\ t \to \infty, \tag{17}$$

where $a_1,\ a_2 > 0,\ \xi_1 > 0$, and the learning cost function approaches its optimal value within the following upper-bound

$$\|Q(\widehat{\boldsymbol{\mu}}) - Q(\boldsymbol{\mu}^{\mathrm{opt}})\| \leq \xi_2(\tfrac{\xi_1}{\omega} + \sqrt{a_1^2 + a_2^2}), \tag{18}$$

as $t \to \infty$, where $\xi_2 = \max_{\boldsymbol{\mu} \in \mathcal{N}(\boldsymbol{\mu}^{\mathrm{opt}})} \|\nabla_{\boldsymbol{\mu}} Q(\boldsymbol{\mu})\|$.

Proof 2: The proof has been removed due to space constraints, but will appear in a longer version of this work.

IV. Main result 3: The case of 3D Boussinesq equation

As an example application of our approach, we consider the 3D incompressible Boussinesq equations that describe the evolution of velocity $\mathbf{v}$, pressure p, and temperature T of a fluid. The coupled equations reflect the conservation of momentum, mass, and energy,

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \nabla \cdot \tau(\mathbf{v}) + \rho \mathbf{g}, \tag{19}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{20}$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = \nabla\left(\kappa \nabla T\right), \tag{21}$$

where the buoyancy force is driven by changes in density $\rho = \rho_0 + \Delta\rho$ and is modeled using the perfect gas law $\Delta\rho\mathbf{g} \approx -\rho_0 \beta \left(T - T_0\right) \mathbf{g},\ \beta = 1/T_0$. The viscous stress $\tau(\mathbf{v}) = \rho\nu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$ with kinematic viscosity ν and thermal conductivity κ. One typically non-dimensionalizes these equations depending on the application at hand. For this study, we perform non-dimensionalization as follows. By introducing a characteristic length L, characteristic velocity v_0, and defining $\tilde{x} = \frac{x}{L}$, $\tilde{t} = \frac{t v_0}{L}$, $\tilde{\mathbf{v}} = \frac{\mathbf{v}}{v_0}$, $\tilde{p} = \frac{p}{\rho v_0^2}$, and $\tilde{T} = \frac{T - T_0}{T_w - T_0}$ we can reduce the number of free parameters to three. These are the Reynolds number $\mathrm{Re} = \frac{\rho v_0 L}{\mu} = \frac{v_0 L}{\nu}$, the Grashof number $\mathrm{Gr} = \frac{g\beta(T_w - T_0)L^3}{\nu^2}$, and the Prandtl number $\mathrm{Pr} = \frac{\nu}{k/\rho c_p}$. Thus,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \tau(\mathbf{v}) + \frac{\mathrm{Gr}}{\mathrm{Re}^2} T, \tag{22}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{23}$$

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \nabla \cdot \left(\frac{1}{\mathrm{RePr}} \nabla T\right), \tag{24}$$

where $\tau(\mathbf{v}) = \frac{1}{\mathrm{Re}}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ and we have dropped the $\tilde{}$s.

Following a Galerkin projection onto the subspace spanned by the POD basis functions, the Boussinesq equation is reduced to a POD ROM with the following structure, e.g., [9]

$$\dot{q}(t) = \mu\ D\ q(t) + [Cq(t)]q(t), \tag{25}$$

$$\mathbf{v}(x, t) = \mathbf{v}_0(x) + \sum_{i=1}^{r_{\mathbf{v}}} q_i(t) \phi_i^{\mathbf{v}}(x), \tag{26}$$

$$T(x, t) = T_0(x) + \sum_{i=r_{\mathbf{v}}+1}^{r_T + r_{\mathbf{v}}} q_i(t) \phi_i^T(x), \tag{27}$$

where $\mu > 0$ is the viscosity, i.e., the inverse of the Reynolds number, D is a negative definite matrix with diagonal blocks corresponding to the viscous stress and thermal diffusion (scaled by Pr to extract the parameter μ) and C is a three-dimensional tensor corresponding to the convection terms in (22) and (24). We notice that this POD-ROM has mainly a linear term and two quadratic terms, so that it can be written in the form (11), with

$$\widetilde{F}(q) = [Cq]q.$$

If we consider bounded parametric uncertainties ΔC for the entries of C, we can write

$$\widetilde{F}(q) = [(C + \Delta C)q]q,$$

where $\|C + \Delta C\|_F \leq c_{\max}$, we have the upper-bound

$$\|\widetilde{F}(q)\| \leq \tilde{f}(q) \equiv c_{\max}\|q\|^2.$$

In this case the nonlinear closure model (14) is

$$H(q) = \mu_{nl} c_{\max} \|q\|^2 \mathrm{diag}(d_{11}, ..., d_{rr}) q, \tag{28}$$

for $\mu_{nl} > 0$ with $d_{ii},\ i = 1, ..., r$ being the diagonal elements of D.

A. Boussinesq equation MES-based POD ROM stabilization

We consider the Rayleigh-Bénard differential-heated cavity problem, modeled with the 3D Boussinesq equation (19) with the following parameters and boundary conditions. The unit cube was discretized with 495k quadratic tetrahedral elements with 611k nodes leading to 1.83M velocity degrees of freedom and 611k temperature degrees of freedom. Thus, $n \approx 2.4 \times 10^6$. The velocity was taken as zero on the boundary and the temperature was specified at ± 0.5 on the x-faces and taken as homogeneous Neumann on the remaining faces. The non-dimensional parameters were taken as $\mathrm{Re} = 4.964 \times 10^4$, $\mathrm{Pr} = 0.712$, and $\mathrm{Gr} = 7.369 \times 10^7$, reasonable values in a quiet room. The simulation (by a finite elements CFD code) was run from zero velocity and temperature and snapshots were collected for 78 seconds.

We apply the results of Theorem 1 and Lemma 1 to this problem. In this case we use 8 POD basis functions for each variable, for the POD model (POD-ROM-G). The upper bounds on the uncertainties in the matrix and tensor entries are assumed to be $c_{\max} = 10$. The two closure model amplitudes $\widehat{\boldsymbol{\mu}} = [\mu_e, \mu_{nl}]^*$ are tuned using the discrete version of the MES algorithm (16), given by

$$\begin{aligned} y_1(k+1) &= y_1(k) + a_1 \Delta t \sin\left(\omega_1 k \Delta t + \frac{\pi}{2}\right) Q(\widehat{\boldsymbol{\mu}}), \\ \widehat{\mu}_e(k+1) &= y_1(k+1) + a_1 \ \sin\left(\omega_1 k \Delta t - \frac{\pi}{2}\right), \\ y_2(k+1) &= y_2(k) + a_2 \Delta t \sin\left(\omega_2 k \Delta t + \frac{\pi}{2}\right) Q(\widehat{\boldsymbol{\mu}}), \\ \widehat{\mu}_{nl}(k+1) &= y_2(k+1) + a_2 \ \sin\left(\omega_2 k \Delta t - \frac{\pi}{2}\right), \end{aligned} \tag{29}$$

where $y_1(0) = y_2(0) = 0$, $k = 0, 1, 2, ...$ is the number of learning iterations, and Δt is the time increment. We use MES parameter values: $a_1 = 0.08$ $[-]$, $\omega_1 = 10$ $[\frac{\mathrm{rad}}{\mathrm{sec}}]$, $a_2 = 10^{-7}$ $[-]$, $\omega_2 = 50$ $[\frac{\mathrm{rad}}{\mathrm{sec}}]$. The learning cost function is chosen as

$$Q(\boldsymbol{\mu}) = \int_0^{t_f} \langle e_T, e_T \rangle_{\mathcal{H}} dt + \int_0^{t_f} \langle e_{\mathbf{v}}, e_{\mathbf{v}} \rangle_{(\mathcal{H})^3} dt. \tag{30}$$

Moreover, $e_T = P_r T_n - T_n^{pod}$, $e_{\mathbf{v}} = P_r \mathbf{v}_n - \mathbf{v}_n^{pod}$ define the errors between the projection of the true model solution onto the POD space $\mathcal{Z}^r$ and the POD-ROM solution for temperature and velocity, respectively.

We first report in Figures 1, and 2 the errors between the true solutions and the POD ROM-G solutions.

Next, we show the learning cost function in Figure 5. We can see a quick decrease of the cost function within the first 20 iterations. This means that the MES manages to improve the overall solutions of the POD ROM quickly. The values of the tuning parameters $\hat{\mu}_e$ and $\hat{\mu}_{nl}$ of the closure model are reported in Figures 6, and 7. Even though the cost function decreases quickly, the ES algorithm continues to fine-tune the parameters $\hat{\mu}_e$, $\hat{\mu}_{nl}$ in the following iterations. The optimal values are then $\hat{\mu}_e \simeq 0.85$, and $\hat{\mu}_{nl} \simeq 1.25 \times 10^{-6}$. We next show the effect of closure-modeling on the accuracy of the POD ROM solutions. Figures 3, 4 show the error of the POD solution with respect to the finite element solution, which by comparison with Figures 1, 2 show an improvement of the POD ROM solutions with the MES tuning of the closure models' amplitudes.

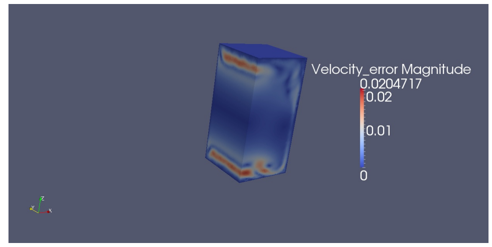

Fig. 1. ROM-G velocity error profile

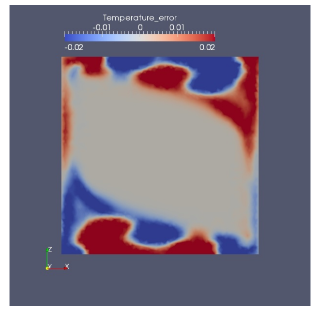

Fig. 2. ROM-G temperature error profile

V. Conclusion

We proposed a new closure model for robust stabilization of reduced order models of PDEs based on a data-driven extremum seeking algorithm to auto-tune the closure model coefficients. We have validated the proposed method on a challenging 3D Boussinesq equation in the form of a modified Rayleigh-Bénard differential-heated cavity problem. The proposed closure model produced accurate and stable reduced order models for the considered test problem. However, future investigations will be conducted on more challenging flows, e.g., turbulent flows, as well as on experimental tests on a water-tank test-bed.

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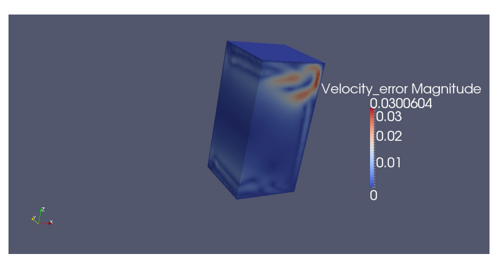

Fig. 3. ROM-G-Learning velocity error profile

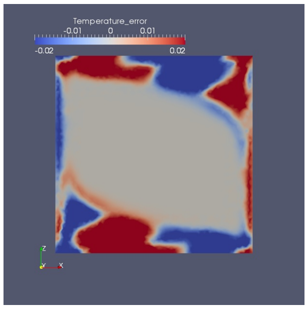

Fig. 4. ROM-G-Learning temperature error profile

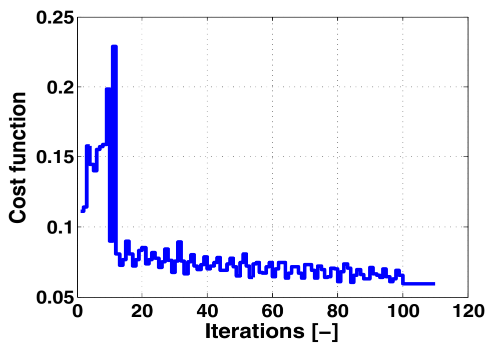

Fig. 5. Learning cost function vs. number of learning iterations

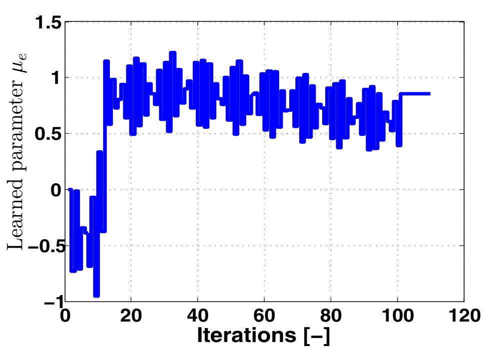

Fig. 6. Coefficient μ_e vs. number of learning iterations

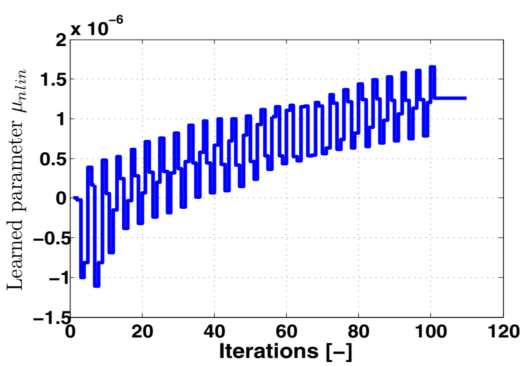

Fig. 7. Coefficient μ_{nl} vs. number of learning iterations

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