2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

Model Reduction for Control of a Multiphysics System: Coupled Burgers' Equation

Boris Kramer

Abstract— **We consider control and design for coupled, multiphysics systems governed by partial differential equations (PDEs). The numerical solution of the control problem involves large systems of ordinary differential equations arising from a spatial discretization scheme, which can be prohibitively expensive. Utilizing reduced order surrogate models evolved as a way to circumvent this computational problem. While many reduced order models work well for simulation, the task of control adds additional complexity. We investigate the effects of different reduced order models on the optimal feedback control. We propose to use a structure-preserving surrogate model, constructed by computing dominant subspaces for each physical quantity separately. This method addresses the different scaling of variables commonly found in multiphysics problems. As a test example, a coupled Burgers' equation multiphysics PDE model is considered. In the numerical study, we find that the feedback gains obtained from the standard proper orthogonal decomposition for the combined variables fail to converge, while the physics-based method produces convergent control feedback matrices.**

I. Introduction

Coupled systems are ubiquitous in engineering problems, as they arise in micro-electromechanical systems [1], lithium-battery models [2], circuit-device systems [3] and fluid dynamics [4]. Simulation of such systems provides valuable insights to the design and control engineer, while often being cheaper and faster to access than experimental results. Complex systems can have multiple physical quantities interacting based on the fundamental laws of physics, e.g., a fluids' temperature and velocity are coupled through buoyancy effects. We consider coupled partial differential equation (PDE) models, which give rise to large, structured systems of ordinary differential equations, by virtue of discretization, e.g., through the finite element method (FEM). Using such large models for design and control can be prohibitive, given time constraints (real-time control) and/or the need for repeated evaluation of models at various parameters (design).

Model order reduction emerged as an important tool to reduce the *computational* complexity of the original high-fidelity model. In essence, model reduction yields computationally cheaper surrogate models, which approximate the original system with respect to a specified criteria. The focus of this paper is on projection-based model reduction techniques derived by Galerkin-projection of the high fidelity model onto a suitable set of modes. An entire toolbox of projection-based reduced order modeling (ROM) methods is available, each producing a different set of modes: balanced truncation, proper orthogonal decomposition, dynamic mode decomposition, balanced POD, reduced basis methods, and Krylov subspace methods. An excellent overview of model order reduction, and the previously mentioned methods, can be found in [5].

Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA 02139. E-mail: bokramer@mit.edu

For certain engineering tasks, it can be sufficient to consider only the input-to-output behavior of a system, in which case Krylov subspace methods provide optimal approximation guarantees [6]. Finding an optimal linear feedback law, however, requires crucial insight into the state-space formulation of the problem. In feedback control, one not only requires that solutions to the open loop system are approximated properly by the surrogate model, but also that the feedback control matrices -or gains- obtained from the reduced model approximate the full feedback well. Hence, we consider methods that focus on accurate state-space approximation, and investigate their feasibility to be used as surrogates in control feedback design. When coupled systems are to be used in control design, we demonstrate that it is advantageous to retain the physics-imposed structure of the original system in the reduced order model. This in turn yields a structure in the feedback matrices, which we show to be important for convergence of the feedback law. We believe that this aids the control community with a deeper understanding of the effects of reduced order models on control.

For ease of illustration, a one dimensional, multiphysics coupled Burgers' PDE model is introduced in §II, together with a finite element discretized model. A short section on reduced order models and POD follows in §III, together with a presentation of the proposed physics-based reduced oder model strategy. In §IV, we present our numerical findings regarding approximation quality for control and simulation of both reduced order models. In particular, we show that by standard projection based model reduction, stability of the model is lost, and the feedback gains fail to converge.

In the remainder of the paper, we use $\dot{f} = \frac{df}{dt}$ as a short notation for the time derivative of $f(t)$, and $f_x(\cdot, x)$ as the partial derivative $\frac{\partial f}{\partial x}(\cdot, x)$. Moreover, the space $L^2((0,\infty);\Omega) := \{f(t,x) : \int_\Omega |f(t,x)|^2 dx < \infty, \ \forall t \in (0,\infty)\}$ and additionally $H^k((0,\infty);\Omega) := \{f \in L^2((0,\infty);\Omega) : \frac{\partial f}{\partial x^k} \in L^2((0,\infty);\Omega)\}$. The transpose of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by A^T, and the Euclidian norm of a vector $x \in \mathbb{R}^n$ is defined as $||x||_2^2 := \sum_{i=1}^n x_i^2$. Sometimes, we shall use `Matlab` short notation for the truncation of matrices and vectors, i.e., $A(j:k, j:k), j < k$ denotes the submatrix of A taken from row (resp. colum) j to k.

978-1-4673-8681-4/$31.00 ©2016 AACC

II. MODELLING, DISCRETIZATION AND CONTROL

A. Partial Differential Equation Model

The coupled Burgers' equation [7] is a one dimensional model that incorporates many interesting questions related to thermal fluid dynamics, commonly modeled by the two or three dimensional Boussinesq equations. The model reads as

$$w_t(t,x) + w(t,x)w_x(t,x) = \mu w_{xx}(t,x) - \kappa T(t,x), \quad (1)$$
$$T_t(t,x) + w(t,x)T_x(t,x) = cT_{xx}(t,x) + b(x)u(t), \quad (2)$$

for $t > 0$ on the one dimensional domain $\Omega = (0,1)$ with boundary conditions

$$w(t,0) = 0 \qquad w_x(t,1) = 0, \quad (3)$$
$$T(t,0) = 0 \qquad T(t,1) = 0, \quad (4)$$

for all $t > 0$ and initial conditions

$$w(0,x) = w_0(x), \quad T(0,x) = T_0(x) \quad \in L^2(\Omega). \quad (5)$$

Here, $w \in H^2((0,\infty);\Omega)$ is a velocity-like function and $T \in H^2((0,\infty);\Omega)$ is a temperature-like function. The parameter κ denotes the coefficient of the thermal expansion, c is the thermal diffusivity, and μ is the viscosity coefficient. The function $b(x)$ specifies the location of the control action $u(t)$, and hence we have a distributed control acting on the temperature, which adversely controls the velocity through the coupling. As a notational remark, $u(t)$ denotes any open loop control, disturbance or excitation, and by $u^*(t)$ we denote the unique solution to the optimal control problem introduced below.

To design a linear feedback controller for the coupled system (1)-(2), a first step consists of linearizing the model around its steady state solution. It is known [8] that the only equilibrium to Burgers' equation with homogeneous, mixed Dirichlet-Neumann boundary conditions (3) is the zero solution. This solution is globally asymptotically stable. By imposing zero Dirichlet boundary conditions on equation (2), the energy eventually dissipates and the system converges uniformly to the zero steady state, independent of the initial condition. Thus, $w_{ss} = T_{ss} = 0$ is an equilibrium to (1)-(2). To this end, the velocity and temperature are decomposed into a steady state and fluctuation part as

$$w(t,x) = w_{ss}(x) + \tilde{w}(t,x) = \tilde{w}(t,x),$$
$$T(t,x) = T_{ss}(x) + \tilde{T}(t,x) = \tilde{T}(t,x).$$

Assuming that the fluctuations are small in the $H^1((0,\infty);\Omega)$ norm, implies $\tilde{w} \cdot \tilde{w}_x \approx 0$ and $\tilde{w} \cdot \tilde{T}_x \approx 0$. Thus, the linearized coupled Burgers' system is given by

$$\dot{\tilde{w}}(t,x) = \mu \tilde{w}_{xx}(t,x) - \kappa \tilde{T}(t,x), \quad (6)$$
$$\dot{\tilde{T}}(t,x) = c\tilde{T}_{xx}(t,x) + b(x)u(t). \quad (7)$$

B. Finite Element Discretization

We discretize the PDE model in space via the finite element method (FEM) with piecewise linear basis functions for both velocity and temperature; they can be replaced by other FE basis functions without changing the exposition below. The finite dimensional system is obtained through Galerkin projection of the fluctuation functions $\tilde{w}(t,x), \tilde{T}(t,x)$ onto the finite element spaces. The coupled Burgers' equation then takes the form of a linear time-invariant system (LTI) with a mass matrix

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (8)$$
$$Ex(0) = x_0 \ \in \mathbb{R}^n, \quad (9)$$

where $x(t) = [x_1^T(t), \ x_2^T(t)]^T$ is the combined state of velocity and temperature coefficients. The parameter $n_1 + n_2 = n$ denotes the size of the combined FEM state space, and the solution snapshots can be partitioned, as

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad (10)$$

where $X_1 \in \mathbb{R}^{n_1 \times s}$ represents the solutions of the velocity, and $X_2 \in \mathbb{R}^{n_2 \times s}$ denotes the part of the system data corresponding to the temperature. We note in passing that the functions $w(\cdot,\cdot)$ and $T(\cdot,\cdot)$ have different scaling and physical meanings. The mass, system, and control input matrices E, A, B have block structure:

$$E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}. \quad (11)$$

Similarly, discretization of the full nonlinear system (1)-(4) reads as

$$E\dot{x}(t) = Ax(t) + F(x(t)) + Bu(t), \quad (12)$$

which is used to generate data for the model reduction step.

C. Optimal Feedback Control

The goal of the optimal control problem is to minimize a specified cost (depending on the state x and control u) subject to dynamic constraints. For the linear quadratic regulator (LQR) optimal control problem, we set the cost

$$J(x(\cdot),u(\cdot)) = \int_0^\infty x(t)^T Ex(t) + u(t)^T Ru(t)dt, \quad (13)$$

and the dynamic constraints are given by the linearized, finite dimensional model (8)-(9). One of many appeals to using an LQR controller is that the optimal control is given by linear feedback [9, p.237f] as

$$u^*(t) = -Kx(t), \quad (14)$$

which exponentially stabilizes the system. The constant *gain matrix* K is constructed as

$$K = R^{-1}B^TPE, \quad (15)$$

where $P \in \mathbb{R}^{n\times n}$ is the unique positive definite solution the algebraic Riccati equation

$$A^TPE + E^TPA - E^TPBR^{-1}B^TPE + C^TC = 0. \quad (16)$$

For large n, solving the above matrix equation is time consuming, as the complexity scales cubicly with the dimension of the state space n. Moreover, reduced order models are often at hand for simulations, and can be used for design and control as well.

III. MOR FOR COUPLED SYSTEMS

We compare two different strategies to obtain reduced order models for the previously motivated multiphysics problem, each having their justification in the toolbox of reduced order modeling techniques. The first strategy yields proper orthogonal modes with an optimal approximation property for the data X. The second method focuses on the physics of the problem, and computes modes from the data X_1, X_2 separately. We refer to [10], [11], [12] for further applications and methods of MOR for coupled systems.

A. Proper Orthogonal Decomposition (POD)

Henceforth, we use POD as a dimension reduction technique within a Galerkin-projection framework, and note that this method can be replaced by any other modal expansion or projection based method for nonlinear model reduction. We refer the reader to Volkwein's notes [13] for theoretical results, an excellent list of references, and implementation details for POD. To begin with, assume a dynamical system of the form

$$E\dot{x}(t) = f(x(t)), \qquad x \in \mathbb{R}^n,$$

is given, where $E \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Solutions at s *snapshot* locations, $x_k := x(t_k)$ for $k = 1, \ldots, s$ are obtained, and stored in the *snapshot matrix*

$$X = [x_1 \; x_2 \ldots \; x_s] \; \in \mathbb{R}^{n\times s}. \tag{17}$$

Let the weighted inner product in $\mathbb{R}^n$ be defined via $(x, y)_E := x^T E y$. Proper orthogonal decomposition provides a basis that optimally represents the given solution data X in the least squares sense, consequently solving the optimization problem

$$\min_{\phi_i} \sum_{j=1}^{s} \left\| x_j - \sum_{i=1}^{r} (x_j, \phi_i)\phi_i \right\|_E^2 \quad \text{s.t.} \quad (\phi_i, \phi_j)_E = \delta_{ij}. \tag{18}$$

We consider the case when $s < n$, i.e. when the data matrix has fewer columns than rows, which leads to the *method of snapshots* for efficient POD computations. We shall see that this only requires computing a singular value decomposition of a square matrix of size $s\times s$. In most complex applications, such as fluid dynamics considered herein, the state space is of enormous size ($n > 10^6$), and the above assumption naturally holds. Solving the optimization problem (18) above with the weighted inner product [13, Ch.1] requires solving

$$E^{1/2}X(E^{1/2}X)^T = \bar{\Phi}\Lambda\bar{\Phi}^T \quad \in \mathbb{R}^{n\times n},$$

which, however, can be replaced by the computationally cheaper decomposition

$$X^T E X = \bar{\Xi}\Lambda\bar{\Xi}^T \quad \in \mathbb{R}^{s\times s}, \tag{19}$$

as we shall see below. In both cases $\Lambda = \Sigma^2$ is the diagonal matrix containing the POD eigenvalues. Consequently, $\bar{\Xi}$ contains the eigenvectors of $X^T E X$ and the columns of $\bar{\Phi}$ are the eigenvectors of $E^{\frac{1}{2}}X(E^{\frac{1}{2}}X)^T$, which are sought. To illustrate the key step of the method of snapshots, let the singular value decomposition of the data be

$$E^{\frac{1}{2}}X = \bar{\Phi}\Sigma\bar{\Xi}^T,$$

so that $\bar{\Phi}, \bar{\Xi}$ are orthogonal (in the non-weighted inner product) and its respective columns satisfy

$$E^{\frac{1}{2}}X\bar{\xi}_i = \sigma_i\bar{\phi}_i, \quad (E^{\frac{1}{2}}X)^T\bar{\phi}_i = \sigma_i\bar{\xi}_i,$$

for $i = 1, \ldots, \text{rank}(X)$. Since $E^{\frac{1}{2}}X\bar{\Xi} = \bar{\Phi}\Lambda^{\frac{1}{2}}$ we have

$$E^{-\frac{1}{2}}\bar{\Phi} = X\Xi\Lambda^{-\frac{1}{2}},$$

and the E-orthogonal *POD modes* that solve the POD minimization problem (18) are then given by [13, Thm 1.3.2]

$$\Phi := E^{-\frac{1}{2}}\bar{\Phi}.$$

With the proper orthogonal modes, the solution can be approximated as

$$x(t) \approx \sum_{i=1}^{r} \hat{x}_i(t)\phi_i$$

which is used in a Galerkin-projection framework to obtain a model of reduced order $r \ll n$.

Remark 1: In multiphysics problems, the scaling of entries in X often varies by several magnitudes, the computation of the product $X^T E X$ or $E^{\frac{1}{2}}X(E^{\frac{1}{2}}X)^T$ can introduce numerical errors. Instead, a direct singular value decomposition of X should be used. Alternatively, subtraction of the column-wise mean of X from the data, or preconditioning, can remedy the scaling problem.

B. Standard Model Order Reduction

The standard projection based model reduction technique is based on optimal modes for the overall data X. The snapshot matrix X consists of solutions to the full coupled dynamics (12) with initial conditions (9). From X, we compute the r proper orthogonal basis functions with the method of snapshots. The reduced order model is obtained through projection onto the modes, see Algorithm 1.

Algorithm 1 : Standard POD Algorithm (STD-POD)

Input: Data X; Model E, A, B, x_0; reduced model order r;
Output: ROM: $\tilde{A}_r, \tilde{B}_r, \tilde{x}_{0,r}$.
1: $X^T E X = \Xi\Lambda\Xi^T$.
2: $\Lambda_r = \Lambda(1:r, 1:r)$; $\Xi_r = \Xi(:, 1:r)$.
3: $\tilde{\Phi} = X\Xi_r\Lambda_r^{-1/2} \in \mathbb{R}^{n\times r}$.
4: $\tilde{A}_r = \tilde{\Phi}^T A\tilde{\Phi}, \quad \tilde{B}_r = \tilde{\Phi}^T B, \quad \tilde{x}_{0,r} = \tilde{\Phi}^T x_0$.

A reduced order model of the linearized dynamics is obtained through Galerkin projection as

$$\text{(STD-POD)} \quad \left\{ \begin{array}{lcl} \dot{\tilde{x}}_r(t) & = & \tilde{A}_r\tilde{x}_r(t) + \tilde{B}_r u(t) \\ \tilde{x}_r(0) & = & \tilde{x}_{0,r} \end{array} \right\},$$

where $E_r = I_r = \tilde{\Phi}^T E\tilde{\Phi}$, since the singular vectors are E - orthonormal. Due to the Schmidt-Eckardt-Young-Mirsky theorem [5, p.37], the above basis $\tilde{\Phi}$ is the best order r basis to represent the snapshot data in the weighted least squares sense. Nonetheless, the coefficients $\tilde{x}_r(t)$ do not enjoy a direct physical interpretation anymore.

C. Physics-Preserving Model Reduction

To preserve the physics, and hence the mathematical (block-) structure of the original system, which then allows for a sound physical interpretation of the corresponding reduced order model coefficients $x_r(t)$, a different projection matrix is constructed. In the case of the Boussinesq equations, similar approaches have been used for simulation [14], [4], [15], but the focus here is on the effect of model reduction on the optimal feedback matrix. From the data X_1, X_2 in (10), separate basis $\hat{\Phi}_1 \in \mathbb{R}^{n_1 \times r_1}$ and $\hat{\Phi}_2 \in \mathbb{R}^{n_2 \times r_2}$ for the velocity and temperature are computed via POD, respectively, and the overall projection matrix is assembled block-wise as $\hat{\Phi} = \text{diag}(\hat{\Phi}_1, \hat{\Phi}_1)$. The projected reduced order model reads as

$$\begin{aligned}
\hat{A}_r &= \begin{bmatrix} \hat{\Phi}_1 & 0 \\ 0 & \hat{\Phi}_2 \end{bmatrix}^T \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \hat{\Phi}_1 & 0 \\ 0 & \hat{\Phi}_2 \end{bmatrix}, \\
&= \begin{bmatrix} \hat{\Phi}_1^T A_1 \hat{\Phi}_1 & \hat{\Phi}_1^T A_{12} \hat{\Phi}_2 \\ 0 & \hat{\Phi}_2^T A_2 \hat{\Phi}_2 \end{bmatrix} = \begin{bmatrix} A_{r_1} & A_{r_{12}} \\ 0 & A_{r_2} \end{bmatrix}, \\
\hat{B}_r &= \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix}^T B = \begin{bmatrix} 0 \\ \Phi_2^T B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ B_{r_2} \end{bmatrix}, \\
\hat{x}_{r,0} &= \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix}^T x_0 = \begin{bmatrix} \hat{x}_{r_1}(0) \\ \hat{x}_{r_2}(0) \end{bmatrix} \quad \in \mathbb{R}^{r_1 + r_2}.
\end{aligned}$$

The mass matrix of the reduced order model is the identity matrix, by virtue of E-orthonormality of the POD basis functions. Let $\hat{x}_r(\cdot) = [\hat{x}_{r_1}(\cdot),\ \hat{x}_{r_2}(\cdot)]^T$ be the state variable of the POD system. Then, the POD-ROM of the linear system (8),(9), is given by

$$(\text{PB-POD}) \quad \left\{ \begin{matrix} \dot{\hat{x}}_r(t) & = & \hat{A}_r \hat{x}_r(t) + \hat{B}_r u(t) \\ \hat{x}_r(0) & = & \hat{x}_{0,r} \end{matrix} \right\}.$$

The structure of this system is identical to (8)-(11). In particular, the coefficients $\hat{x}_{r_1}(\cdot)$ and $\hat{x}_{r_2}(\cdot)$ correspond to the evolution of velocity, and temperature in the reduced space, respectively. Another advantage of this approach is that it allows for an independent choice of the order of the surrogate models r_1 and r_2. For instance, the temperature T might be easier to approximate than the velocity, so $r_1 \leq r_2$. The physics-based reduced order model procedure is summarized in Algorithm 2.

D. Feedback Control via ROMs

We circumvent the expensive computation of the Riccati equation (16) by incorporating surrogate models. Therefore, we compute P_r, the solution of the low order ARE

$$P_r A_r + A_r^T P_r - P_r B_r R^{-1} B_r^T P_r + C_r^T C_r = 0, \qquad (20)$$

and project the feedback matrix $K_r = R^{-1} B_r^T P_r$ from the r-dimensional POD subspace up to the n-dimensional FE space via

$$K \approx K_r^n := \Phi K_r,$$

Algorithm 2 : **Physics-Based POD Algorithm** (PB-POD)

Input: Data $X1, X2$; Model $E1, E2, A, B, x_0$; reduced model order r;

Output: ROM: $A_r, B_r, x_{0,r}$.

$X = [X_1,\ X_2]$ according to physics variables.
$X_1^T E_1 X_1 = \Xi_1 \Sigma_1 \Xi_1^T$.
$X_2^T E_2 X_2 = \Xi_2 \Sigma_2 \Xi_2^T$.
$\Lambda_{1,r_1} = \Lambda_1(1:r_1, 1:r_1)$; $\Xi_{1,r_1} = \Xi_1(:, 1:r_1)$.
$\Lambda_{2,r_2} = \Lambda_2(1:r_2, 1:r_2)$; $\Xi_{2,r_2} = \Xi_2(:, 1:r_2)$.
$\hat{\Phi}_1 = X_1 \Xi_{1,r_1} \Lambda_{1,r_1}^{-1/2} \in \mathbb{R}^{n_1 \times r_1}$.
$\hat{\Phi}_2 = X_2 \Xi_{2,r_2} \Lambda_{2,r_2}^{-1/2} \in \mathbb{R}^{n_2 \times r_2}$.
$\hat{\Phi} = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \in \mathbb{R}^{n \times r},\ n = n_1 + n_2,\ r = r_1 + r_2.$
$\hat{A}_r = \hat{\Phi}^T A \hat{\Phi}, \quad \hat{B}_r = \hat{\Phi}^T B, \quad \hat{x}_{0,r} = \hat{\Phi}^T x_0.$

where $\Phi = \tilde{\Phi}$ yields $\tilde{K}_r^n$ for (STD-POD), and $\Phi = \hat{\Phi}$ produces the gain $\hat{K}_r^n$ for (PB-POD).

Remark 2: In future research, we envision using black-box simulation code for the above methods in a matrix-free setup. Therefore, we restrict ourselves to using only simulation data for construction of the reduced order models, and no dual information, such as in balanced truncation and its variants [16], [17]. This necessitates the careful investigation of the control properties of the reduced order models, as done in the next section.

IV. Numerical Results

For the coupled Burgers' equation, the viscosity is set to $\mu = 10^{-3}$, and the parameters $\kappa = 1.0$, $c = 10^{-2}$; the system is simulated for $5s$ and 100 snapshots are recorded at equidistant times. A control penalty is set to $R = 0.1$ in (13). The initial conditions (5) are $w_0(x) = 2\left(x^2\left(\frac{1}{2} - x\right)^2\right)$ and $T_0(x) = 5\sin\left(\frac{1}{2}x\right)$. We mention in passing that the feedback gains obtained from the finite element method converged quickly with increasing n. Thus, we choose $n = 128$ as the discretization order of the high fidelity ("truth") model. With the above specifications, we simulate the nonlinear finite element system (12) with initial conditions (9) and subsequently compute reduced order models from this data through (STD-POD) or (PB-POD).

We first compare the feedback matrices K_r^n from both methods for increasing r as outlined in §III-D. Thus, let $r = r_1 + r_2$ in the standard reduced order model, where r_1, r_2 are the number of POD basis functions chosen for approximation of the velocity and temperature in the physics-based approach, respectively. The gains converge for the physics-based model, see Figures 1 and 2, whereas they fail to converge for the standard POD based reduced order model, see Figures 3-4, and instead grow in magnitude. The reduced order models for (STD-POD) are unstable, e.g., the largest computed reduced model with $r = 20$ still has two eigenvalues of the system matrix $\hat{A}_r$ with positive real part, $\lambda_{1,2} = 0.128 \pm 0.007i$, where i denotes the imaginary unit. All lower order models are unstable, too. The *transfer function* of system (8)-(9) results from a Laplace transform

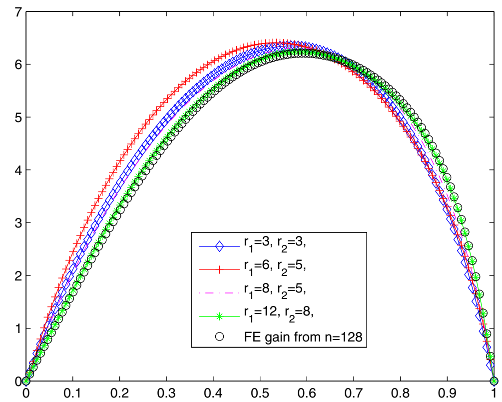

Fig. 1. Feedback gains for temperature $\hat{K}_T^n$ of model (PB-POD).

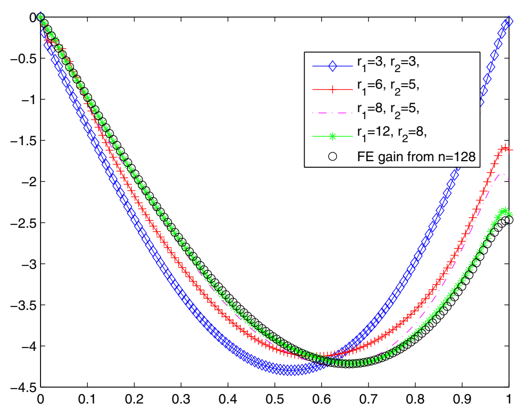

Fig. 2. Feedback gains for temperature $\hat{K}_w^n$ of model (PB-POD).

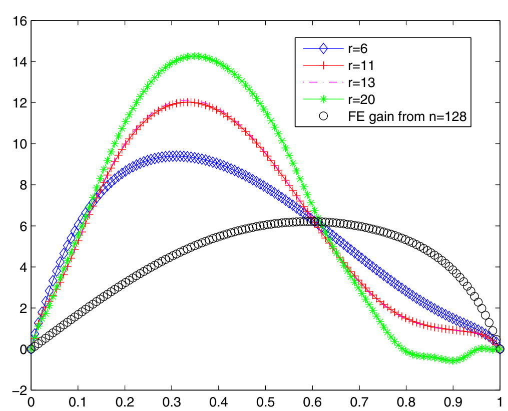

Fig. 3. Feedback gains for temperature $\tilde{K}_T^n$ of model (STD-POD).

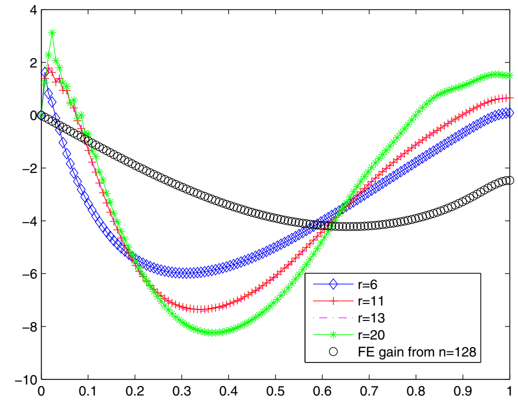

Fig. 4. Feedback gains for temperature $\tilde{K}_w^n$ of model (STD-POD).

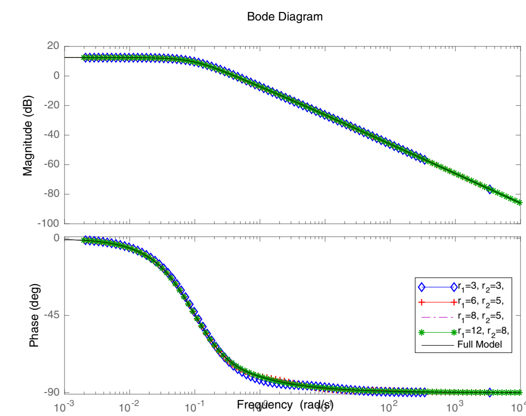

Fig. 5. Bode plot for the physics-based reduced order model (PB-POD).

(time variable t to frequency variable s), and is defined as $G(s) := (sE - A)^{-1}B$. A comparison of the full transfer function G to $\tilde{G}_r(s) := (sI_r - \tilde{A}_r)^{-1}\tilde{B}_r$ and $\hat{G}_r(s) := (sI_r - \hat{A}_r)^{-1}\hat{B}_r$ obtained from the two reduced order models provides insight into the approximation quality of the open loop systems. Figures 5-6 show the Bode plots (transfer function vs. frequency) for both reduced order models. The magnitude and phase of the physics based reduced order models (PB-ROM) match the full order model indisputably well, see Figure 5. From Figure 6 one can see that neither the amplitude, nor phase of the models match for slow frequencies. Yet for higher frequencies, the magnitude of the (STD-POD) reduced order model matches the high-fidelity model, and the phases converge (adding 360 degrees to the plot). Since the error between the original and reduced transfer function bounds the output error in the frequency domain is

$$\|x - x_r\|_2 \leq \|G - G_r\|_{\mathcal{H}_\infty} \|u\|_2,$$

the errors $\|G - G_r\|_{\mathcal{H}_\infty}$ are used to asses the quality of the reduced order models. Here, the (Hardy-) norm of the transfer function is defined as $\|G\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|G(i\omega)\|_2$.

TABLE I

ERROR MEASURES FOR THE STANDARD MODEL REDUCTION, AND PHYSICS-BASED MODEL REDUCTION METHOD.

(r_1, r_2)	$\|\|G - \tilde{G}_r\|\|_\infty$	$\|\|G - \hat{G}_r\|\|_\infty$	$\|\|X - \tilde{\Phi}\tilde{\Phi}^T X\|\|_2$	$\|\|X - \hat{\Phi}\hat{\Phi}^T X\|\|_2$	$\|\|K - K(\tilde{\Phi}\tilde{\Phi}^T)E\|\|_2$	$\|\|K - K(\Phi\Phi^T)E\|\|_2$
(3,3)	8.81×10^{-2}	4.40×10^{0}	5.83×10^{-2}	2.40×10^{-2}	1.16×10^{-1}	1.74×10^{-1}
(6,5)	1.32×10^{-2}	3.98×10^{0}	2.00×10^{-2}	4.09×10^{-3}	2.38×10^{-2}	9.85×10^{-2}
(8,5)	1.32×10^{-2}	3.98×10^{0}	2.00×10^{-2}	1.19×10^{-3}	2.01×10^{-2}	9.81×10^{-2}
(12,8)	3.57×10^{-4}	4.01×10^{0}	4.26×10^{-3}	1.04×10^{-4}	1.80×10^{-2}	4.65×10^{-2}

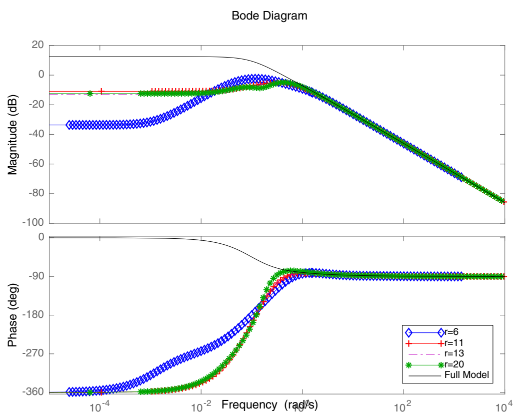

Fig. 6. Bode plot for the standard POD reduced order model (STD-POD).

In Table I, we list the errors in the transfer functions $\|G - G_r\|_{\mathcal{H}_\infty}$. Moreover, we provide insight into how well the computed POD modes for both strategies are suitable to approximate the data ($||X - \Phi\Phi^T X||_2$) and the true feedback matrices ($||K - K(\Phi\Phi^T)E||_2$). As expected from the optimality result of POD for the standard model reduction method (STD-POD), the approximation of the snapshot set is better than model (PB-POD). However, this conclusion reverses as we look at the approximation of the feedback gain functions, where the physics-based model gives better approximation results. This adds to our findings in Figures 1 and 4 above.

V. CONCLUSION

At the current maturity of the field of model reduction, numerous excellent methods exist, and it is up to the engineer/user to decide what suits best for the application at hand. Were we to only focus on a method producing modes for optimal data approximation, valuable information about the physics of the problem gets lost. We advocate that extra effort for constructing structure-preserving reduced order models for control and design of coupled multiphysics systems is beneficial, despite the fact that we loose the optimality guarantee for approximation of the overall snapshot set. We acknowledge that further work regarding guarantees for convergence of reduced order gains to their high-fidelity counterparts is necessary.

ACKNOWLEDGEMENTS

The author would like to thank Professor John A. Burns at Virginia Tech, Professor John R. Singler at Missouri University of Science and Technology, and Dr. Weiwei Hu at the Institute for Mathematics and Its Applications for helpful comments and discussions.

REFERENCES

[1] D. S. Bindel, Z. Bai, and J. W. Demmel, "Model reduction for RF MEMS simulation," in *Applied Parallel Computing. State of the Art in Scientific Computing*. Springer, 2006, pp. 286–295.

[2] O. Lass and S. Volkwein, "POD galerkin schemes for nonlinear elliptic-parabolic systems," *SIAM Journal on Scientific Computing*, vol. 35, no. 3, pp. A1271–A1298, 2013.

[3] M. Hinze, M. Kunkel, A. Steinbrecher, and T. Stykel, "Model order reduction of coupled circuit-device systems," *International Journal of Numerical Modelling: Electronic Networks, Devices and Fields*, vol. 25, no. 4, pp. 362–377, 2012.

[4] S. S. Ravindran, "Error analysis for Galerkin POD approximation of the nonstationary Boussinesq equations," *Numerical Methods for Partial Differential Equations*, vol. 27, no. 6, pp. 1639–1665, 2011.

[5] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems (Advances in Design and Control)*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2005.

[6] S. Gugercin, A. C. Antoulas, and C. A. Beattie, "H2 model reduction for large-scale linear dynamical systems." *SIAM J. Matrix Analysis Applications*, vol. 30, no. 2, pp. 609–638, 2008.

[7] B. Kramer, "Model reduction of the coupled Burgers equation in conservation form," Master's thesis, Virginia Tech, 2011.

[8] E. J. Allen, J. A. Burns, and D. S. Gilliam, "Numerical approximations of the dynamical system generated by Burgers equation with Neumann–Dirichlet boundary conditions," *ESAIM: Mathematical Modelling and Numerical Analysis*, vol. 47, no. 05, pp. 1465–1492, 2013.

[9] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley-Interscience, 1972.

[10] A. Lutowska, "Model order reduction for coupled systems using low-rank approximations," Ph.D. dissertation, PhD thesis, Eindhoven University of Technology, 2012.

[11] T. Reis and T. Stykel, "A survey on model reduction of coupled systems," in *Model order reduction: theory, research aspects and applications*. Springer, 2008, pp. 133–155.

[12] P. Benner and L. Feng, "Model order reduction for coupled problems: Survey," *Applied and Computational Mathematics*, vol. 14, no. 1, pp. 3–22, 2015.

[13] S. Volkwein, "Proper orthogonal decomposition: Theory and reduced-order modelling," Lecture Notes, University of Konstanz, available at http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/scripts.php, 2013.

[14] A. Liakopoulos, P. Blythe, and H. Gunes, "A reduced dynamical model of convective flows in tall laterally heated cavities," in *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 453, no. 1958. The Royal Society, 1997, pp. 663–672.

[15] O. San and J. T. Borggaard, "Basis selection and closure for POD models of convection dominated Boussinesq flows," in *21st International Symposium on Mathematical Theory of Networks and Systems*, 2014, pp. 132–139.

[16] B. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *Automatic Control, IEEE Transactions on*, vol. 26, no. 1, pp. 17–32, 1981.

[17] K. Willcox and J. Peraire, "Balanced model reduction via the proper orthogonal decomposition," *AIAA Journal*, pp. 2323–2330, 2002.