2015 American Control Conference
Palmer House Hilton
July 1-3, 2015. Chicago, IL, USA

Full Flux Models For Optimization and Control of Heat Exchangers

John A. Burns and Boris Kramer

***Abstract*— If convection is the dominate mechanism for heat transfer in a heat exchangers, then the devices are often modeled by hyperbolic partial differential equations. One of the difficulties with this approach is that for low (or zero) pipe flows, some of the imperial functions used to model friction can become singular. One way to address low flows is to include the full flux in the model so that the equation becomes a convection-diffusion equation with a "small" diffusion term. We show that solutions of the hyperbolic equation are recovered as limiting (viscosity) solutions of the convection-diffusion model. We employ a composite finite element - finite volume scheme to produce finite dimensional systems for control design. This scheme is known to be unconditionally L^2-stable, uniformly with respect to the diffusion term. We present numerical examples to illustrate how the inclusion of a small diffusion term can impact controller design.**

I. Introduction

In recent years there has been a flurry of papers on modeling, simulation and control of simple heat exchangers. These models are special cases of more general thermal fluid systems and, although there are a multitude of models for such systems, we focus on simple models described by convection-diffusion equations with a small or zero diffusion term. These models may be found in the recent papers [1], [2], [3] and [4]. We will consider variations of the models in Sano's papers [5], [6] and [7].

In [5], Sano considered a control problem for a mono-tubular heat exchanger which he modeled by the $1D$ hyperbolic control system

$$T_t(t,x) = -vT_x(t,x) - \kappa T(t,x) + \gamma e^{-bx}u(t), \qquad (1)$$

$$0 < x < L, \quad t > 0,$$

with boundary condition

$$T(t,0) = 0, \qquad (2)$$

where $u(\cdot) \in L^2(0,+\infty)$ is a control input. The initial data for (1) is given by

$$T(0,x) = \varphi(x) \in L^2(0,L). \qquad (3)$$

The physical set up involves a fluid of constant density ρ and of heat capacity C_p flowing through an internal tube of a mono-tube heat exchanger of length L. The flow velocity is assumed to be the constant $v > 0$. After some simplifying assumptions, one arrives at equation (1) where $\kappa \geq 0$ plays the role of a heat transfer coefficient and $T(t,x)$ is the temperature of the liquid. The function $b(x) = \gamma e^{-qx}$ distributes the control, but for $q \geq 1$ the control is concentrated near $x = 0$ to approximate a boundary control problem (see [5]). Note that the case $\gamma = 0$ implies $b(x) = 0$ and corresponds to the uncontrolled system.

It is assumed that convection dominates so that diffusion can be ignored However, this presents as problem when the flow velocity becomes small or zero and the hyperbolic equation becomes degenerate. The "full flux" model modifies (1) by adding a diffusion term of the form $\varepsilon T_{xx}(t,x)$ (where $\varepsilon > 0$) and (1) is replaced by

$$T_t(t,x) = \varepsilon T_{xx}(t,x) - vT_x(t,x) - \kappa T(t,x) + \gamma e^{-bx}u(t). \qquad (4)$$

The boundary condition (7) is augmented by a flux condition at $x = L$. In particular, the boundary conditions for (4) are

$$T(t,0) = 0, \quad \varepsilon T_x(t,L) = 0. \qquad (5)$$

The "ε" in the flux boundary condition at $x = L$ is important and should not be dropped since we need this to establish "convergence of the system" (4)-(5) to the system (1)-(2).

Observe that if $\varepsilon = 0$ and $v > 0$, the parabolic system (4)-(5) reduces to the hyperbolic system (1)-(2) and, if $\varepsilon > 0$ and $v = 0$, then (4)-(5) reduces to a self adjoint heat equation. It is easy to show that if $\varepsilon > 0$ and $v > 0$, then $v \to 0^+$ implies that solutions of (4)-(5) converge (in $L^2(0,L)$) to the solutions of

$$T_t(t,x) = \varepsilon T_{xx}(t,x) - \kappa T(t,x) + \gamma e^{-bx}u(t),$$

with boundary conditions

$$T(t,0) = 0, \quad \varepsilon T_x(t,L) = 0.$$

On the other hand, showing that if $\varepsilon > 0$ and $v > 0$, then $\varepsilon \to 0^+$ implies that solutions of (4)-(5) converge to the solutions of (1)-(2) requires more effort and will be addressed below. Finally, assuming one can prove convergence of these systems, the issue of approximation still needs to be addressed. Ideally, one would like to have an approximation scheme that is valid for "small" $\varepsilon > 0$ and "large" $v > 0$ and remains convergent for "small" $\varepsilon > 0$ and "smaller" $v > 0$. Moreover, we need these numerical schemes to be dual convergent if they are to be useful in optimization based design (see [8], [9]). In this paper we employ a composite finite element - finite volume discretization method discussed in [10] and [11]. This scheme is known to be unconditionally L^2-stable, uniformly with respect to the diffusion term.

Because of space limitations, theoretical results are provided only for the mono-tube heat exchanger. However, the results extend to parallel and counter flow heat exchangers and we present numerical results for the counter flow heat exchanger illustrated in Figure 1. In the top channel one

John A. Burns, Interdisciplinary Center for Applied Mathematics, Virginia Tech, Blacksburg, VA 24061 jaburns@vt.edu

Boris Kramer, Interdisciplinary Center for Applied Mathematics, Virginia Tech, Blacksburg, VA 24061 bokr@vt.edu

has a "hot" fluid with temperature $T_1(t,x)$ flowing from left to right with constant velocity v_1 and a "cold" fluid with temperature $T_2(t,x)$ flowing from right to left with constant velocity v_2. The cold fluid is heated by heat transfer through the wall. We consider a control system similar to the problem considered by Sano and Nakagiri in [12] for a parallel flow heat exchanger and may be considered as a two fluid version of mono-tube problem above.

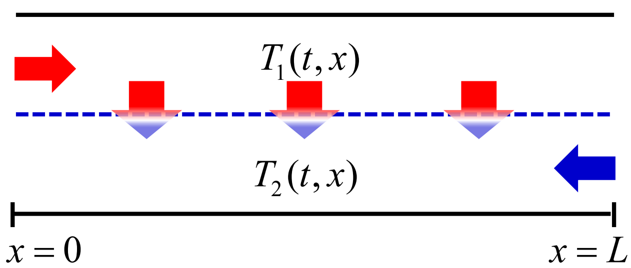

Fig. 1. A Counterflow Heat Exchanger

The full flux model for this system has the form

$$\begin{aligned} \frac{\partial T_1(t,x)}{\partial t} &= \varepsilon \frac{\partial^2 T_1(t,x)}{\partial x^2} - v_1 \frac{\partial T_1(t,x)}{\partial x} - \kappa_1 T_1(t,x) \\ &\quad + \kappa_1 T_2(t,x) + \gamma_1 e^{-b_1 x} u_1(t) \\ \frac{\partial T_2(t,x)}{\partial t} &= \varepsilon \frac{\partial^2 T_2(t,x)}{\partial x^2} + v_2 \frac{\partial T_2(t,x)}{\partial x} - \kappa_2 T_2(t,x) \\ &\quad + \kappa_2 T_1(t,x) + \gamma_2 e^{-b_2 x} u_2(t) \end{aligned}, \quad (6)$$

where the constants κ_1 and κ_2 are heat transfer coefficients. For channel one, we have the boundary conditions

$$T_1(t,0)=0, \quad \varepsilon T_x(t,L)=0 \quad (7)$$

and for channel two we have

$$-\varepsilon T_x(t,0)=0, \quad T_2(t,L)=0. \quad (8)$$

Initial conditions for each channel are given by

$$T_1(0,x)=\varphi(x) \ \text{ and } \ T_2(0,x)=\psi(x), \quad 0<x<L, \quad (9)$$

respectively.

II. Distributed Parameter Formulation

We focus our analysis on the mono-tube full flux system (4)-(5), but note that the results can be extended to parallel and counter flow heat exchanger models such as (6)-(8) above. Note that (4)-(5) may be written as an abstract Cauchy problem on $Z=L^2(0,L)$ of the form

$$\dot{z}(t)=[\varepsilon\mathscr{A}+v\mathscr{H}]z(t)+\mathscr{K}z(t)+\mathscr{B}u(t)\in Z, \quad (10)$$

where

$$\mathscr{D}(\mathscr{A})=\{\varphi(\cdot)\in H^2(0,L):\varphi(0)=0, \quad \varepsilon\varphi'(0)=0\}, \quad (11)$$

$$\mathscr{A}\phi(\cdot)=\phi''(\cdot), \quad (12)$$

$$\mathscr{D}(\mathscr{H})=\{\varphi(\cdot)\in H^1(0,L):\varphi(0)=0\}, \quad (13)$$

$$\mathscr{H}\phi(\cdot)=-\phi'(\cdot), \quad (14)$$

$\mathscr{K}\phi(\cdot)=-\kappa\phi(\cdot)$ and $[\mathscr{B}u](x)=e^{-bx}u$. It is well known that for all $\varepsilon>0$ and $v>0$ the operator $\mathscr{U}=[\varepsilon\mathscr{A}+v\mathscr{H}]+\mathscr{K}$ with domain $\mathscr{D}(\mathscr{U})=\mathscr{D}(\mathscr{A})$ generates an analytic semigroup $S(t,\varepsilon,v)$, the operator $\varepsilon\mathscr{A}+\mathscr{K}$ generates an analytic semigroup $M(t,\varepsilon)$ and $v\mathscr{H}+\mathscr{K}$ generates a nilpotent semigroup $N(t,v)$, all on $Z=L^2(0,L)$ (see [13], [14], [5]). Observe that

$$S(t,\varepsilon,0)=M(t,\varepsilon)$$

and

$$S(t,0,v)=N(t,v).$$

In order to generate useful approximation schemes that are valid for ε and v, we first need to show that the mappings $v\to S(t,\varepsilon,v)$ and $\varepsilon\to S(t,\varepsilon,v)$ are strongly continuous as $v\to 0^+$ and $\varepsilon\to 0^+$. The following result is classical and a proof can be found in standard books (see [15] and [16]).

Theorem 1: For each $\hat{\varepsilon}>0$, The semigroup $S(t,\hat{\varepsilon},v)$ converges strongly to $S(t,\hat{\varepsilon},0)=M(t,\hat{\varepsilon})$ as $v\to 0^+$. Moreover, this convergence is uniform on compact time intervals.

The more interesting case concerns the strong convergence of $S(t,\varepsilon,\hat{v})$ to $S(t,0,\hat{v})=N(t,\hat{v})$ as $\varepsilon\to 0^+$ since this involves what appears to be a singular perturbation result. To prove this result we first consider the boundary value problem

$$-vw_x(x)=f(x), \ 0<x<L, \quad w(0)=0, \quad (15)$$

where $f\in L^2(0,L)$ is given. For $\varepsilon>0$ consider the perturbed system

$$\varepsilon w_{xx}(x)-vw_x(x)=f(x), \quad 0<x<L, \quad (16)$$

with boundary conditions

$$w(0)=0, \quad \varepsilon w_x(L)=0. \quad (17)$$

The solution to (15) is given by

$$w^0(x)=-\int_0^x (f(s)/v)ds=-\int_0^x h(s)ds$$

where $h(s)=(f(s)/v)$. Let

$$g(x)=\int_0^x h(s)ds$$

so that $g(x)=-w^0(x)$. To simplify notation, we set $R=R(\varepsilon)=v/\varepsilon$ so that for a fixed $v=\hat{v}$ we have $R(\varepsilon)\to+\infty$ as $\varepsilon\to 0^+$ and (16) becomes

$$w_{xx}(x)-Rw_x(x)=Rh(x), \quad 0<x<L. \quad (18)$$

Integrating (18) produces

$$w_x(x)-Rw(x)=R\int_0^x h(s)ds+Rc=R[g(x)+c] \quad (19)$$

for some constant $c=c_R$. Since $w(0)=0$, the Variation of Parameters formula yields

$$w(x)=w^R(x)=\int_0^x e^{R(x-s)}R[g(s)+c]ds. \quad (20)$$

Integrating (20) by parts we have

$$\begin{aligned} w^R(x) &= \left[-e^{R(x-s)}(g(s)+c)\right]_{s=0}^{s=x} + \int_0^x e^{R(x-s)} g'(s)ds \\ &= -[g(x)+c] + ce^{Rx} + \int_0^x e^{R(x-s)} g'(s)ds \qquad (21) \\ &= -g(x) + [e^{Rx}-1]c + \int_0^x e^{R(x-s)} h(s)ds. \end{aligned}$$

The boundary condition $\varepsilon w_x^R(L) = 0$ uniquely determines the constant c_R . In particular, (19) implies

$$w_x^R(L) - Rw^R(L) = R[g(L)+c]$$

so that

$$\varepsilon w_x^R(L) - \nu w^R(L) = \nu[g(L)+c]$$

and since $\varepsilon w_x^R(L) = 0$, it follows that $\nu w^R(L) = -\nu[g(L)+c]$ which implies

$$c = -g(L) - w^R(L). \qquad (22)$$

To compute the constant $c = c_R$ we substitute equation (21) into (22) to obtain

$$\begin{aligned} c &= -g(L) - \left\{-g(L) + [e^{RL}-1]c + \int_0^L e^{R(L-s)} h(s)ds.\right\} \\ &= -g(L) + g(L) - [e^{RL}-1]c - \int_0^L e^{R(L-s)} h(s)ds \\ &= -e^{RL}c + c - \int_0^L e^{R(L-s)} h(s)ds. \end{aligned}$$

Consequently,

$$0 = -ce^{RL} - \int_0^L e^{R(L-s)} h(s)ds$$

and solving for c yields

$$c = c_R = \frac{-1}{e^{RL}} \int_0^L e^{R(x-s)} h(s)ds = -\int_0^L e^{-Rs} h(s)ds. \qquad (23)$$

Observe that

$$\begin{aligned} |c_R| &\le \int_0^L \left|e^{-Rs}\right| |h(s)|\, ds \le \left[\int_0^L e^{-2Rs}ds\right] \left[\int_0^L |h(s)|^2\, ds\right] \\ &= \left(\left[\frac{-1}{2R}\left[e^{-2Rs}\right]\right]_{s=0}^{s=L}\right) \|h(\cdot)\|_{L^2}^2 \qquad (24) \\ &= \frac{1}{2R}\left(1-e^{-2RL}\right) \|h(\cdot)\|_{L^2}^2 . \end{aligned}$$

so that

$$\lim_{\varepsilon \to 0} c_R = 0.$$

Regrouping (21), we have

$$w^R(x) + g(x) = [e^{Rx}-1]c_R + \int_0^x e^{R(x-s)} h(s)ds \qquad (25)$$

which we write as

$$w^R(x) - w^0(x) = E(R,x) \qquad (26)$$

with $E(R,x)$ defined by

$$E(R,x) = [e^{Rx}-1]c_R + \int_0^x e^{R(x-s)} h(s)ds. \qquad (27)$$

Lemma 1: The function $E(R,x)$ converges to zero as $\varepsilon \to 0^+$ and the convergence is uniform on the compact interval $[0,L]$.

Proof: Using the definition (27) and (23), we have

$$\begin{aligned} E(R,x) &= [e^{Rx}-1]c_R + \int_0^x e^{R(x-s)} h(s)ds \\ &= -c_R + e^{Rx}c_R + \int_0^x e^{R(x-s)} h(s)ds \\ &= -c_R + e^{Rx}\left[-\int_0^L e^{-Rs} h(s)ds\right] + \int_0^x e^{R(x-s)} h(s)ds \\ &= -c_R - \int_0^L e^{R(x-s)} h(s)ds + \int_0^x e^{R(x-s)} h(s)ds \\ &= -c_R - \int_x^L e^{-R(x-s)} h(s)ds. \end{aligned}$$

However,

$$\begin{aligned} \left|\int_x^L e^{-R(x-s)} h(s)ds\right| &\le \int_x^L e^{-R(x-s)}\, |h(s)|\, ds \\ &\le \left(\int_x^L e^{-2R(x-s)}ds\right) \left[\int_x^L |h(s)|^2\, ds\right] \qquad (28) \\ &\le \frac{1}{2R}\left(1-e^{-2R(L-x)}\right) \|h(\cdot)\|_{L^2}^2 . \end{aligned}$$

and combining (24) and (28) it follows that

$$\begin{aligned} |E(R,x)| &\le |c_R| + \left|\int_x^L e^{-R(x-s)} h(s)ds\right| \\ &\le \frac{1}{R}\left(1-e^{-2RL}\right) \|h(\cdot)\|_{L^2}^2 . \end{aligned}$$

Hence,

$$|E(R,x)| \le \varepsilon\, \|h(\cdot)\|_{L^2}^2$$

and

$$\lim_{\varepsilon \to 0^+} E(R,x) = 0.$$

Clearly the convergence is uniform on $[0,L]$ and this completes the proof of the Lemma. ■

Theorem 2: The solution $w^R(x)$ converses uniformly on $[0,L]$ to $w^0(x)$.

The proof of this theorem is a direct consequence of equation (26) and the previous lemma. We shall use this result to prove the following theorem.

Theorem 3: For each $\hat{\nu} > 0$, the semigroup $S(t,\varepsilon,\hat{\nu})$ converges strongly to $S(t,0,\hat{\nu}) = N(t,\hat{\nu})$ as $\varepsilon \to 0^+$. Moreover, this convergence is uniform on compact time intervals.

Proof: We employ the Trotter-Kato Theorem (see [16]). Let $\hat{\nu} > 0$ be fixed and observe that $S(t,\varepsilon,\hat{\nu})$ and $N(t,\hat{\nu})$ are of class $G(1,-k)$. Thus, by Theorem 4.2 in [16] to establish strong convergence of $S(t,\varepsilon,\hat{\nu})$ to $N(t,\hat{\nu})$ it is sufficient to establish that for $\lambda > -k$, one has resolvent convergence

$$\lim_{\varepsilon \to 0^+} (\lambda I - [(\varepsilon\mathscr{A} + \hat{\nu}\mathscr{H}) + \mathscr{K}])^{-1} f = (\lambda I - [\hat{\nu}\mathscr{H} + \mathscr{K}])^{-1} f$$

for all $f \in Z = L^2(0,L)$. Moreover, it is sufficient to establish this convergence for $\lambda_0 = 0$ (see the remark on page 86 in

Pazy [16]). Therefore, we need to show that for $f \in Z = L^2(0,L)$

$$\lim_{\varepsilon \to 0^+} ([(\varepsilon \mathscr{A} + \hat{v}\mathscr{H}) + \mathscr{K}]^{-1} f = [\hat{v}\mathscr{H} + \mathscr{K}]^{-1} f.$$

Since $\mathscr{K}$ is bounded the problem reduces to proving that

$$\lim_{\varepsilon \to 0^+} (\varepsilon \mathscr{A} + \hat{v}\mathscr{H})^{-1} f = (\hat{v}\mathscr{H})^{-1} f \tag{29}$$

for all $f \in Z = L^2(0,L)$. However, from equation (26) above we have

$$[(\varepsilon \mathscr{A} + \hat{v}\mathscr{H})^{-1} f](x) - [(\hat{v}\mathscr{H})^{-1} f](x) = E(R,x)$$

so that

$$\left| [(\varepsilon \mathscr{A} + \hat{v}\mathscr{H})^{-1} f](x) - [(\hat{v}\mathscr{H})^{-1} f](x) \right| \leq |E(R,x)|$$

and the previous Theorem implies (29) holds and this completes the proof. ■

III. Numerical Approximations of The System

We return now to the full flux version of the controlled system

$$\dot{z}(t) = [\varepsilon \mathscr{A} + v\mathscr{H}]z(t) + \mathscr{K} z(t) + \mathscr{B}u(t) \in Z \tag{30}$$

which we write as

$$\dot{z}(t) = [\mathscr{A}(\varepsilon) + \mathscr{F}(v)]z(t) + \mathscr{B}u(t), \tag{31}$$

where $\mathscr{A}(\varepsilon) = \varepsilon \mathscr{A}$ and $\mathscr{F}(v) = v\mathscr{H} + \mathscr{K}$. The process of numerical approximation can be thought of as a method to produce hierarchal finite dimensional models which in turn can be used for simulation, control and optimization. It is important that these models respect the physics and are constructed so that there is sufficient smoothness in the model and parameters to allow for the implementation of modern solvers.

Of course this system represents the standard convection-diffusion equation and the important parameter is the Péclet number $Pe = v/\varepsilon$. When the Péclet number is large, special numerical methods are required (up-winding, etc.) to accurately simulate the system. We are interested in numerical schemes that produce "good approximations" for $0 \leq Pe < \bar{P}$. The goal is to produce finite dimensional models for the typical heat exchangers described by (10) of the form

$$\dot{z}_N(t) = [\mathscr{A}_N(\varepsilon) + \mathscr{F}_N(v)]z_N(t) + \mathscr{B}_N u(t), \tag{32}$$

which have the following properties:

1) The systems (32) preserve the properties in (10) that are essential for simulation, control design and optimization (smoothness, observability and reachability, etc.).
2) For sufficiently small $\hat{\varepsilon} > 0$, the control system (32) approximates (10) for $0 \leq Pe < \bar{P}$. In particular, the approximate control system (32) is valid for zero flows (i.e., $v = 0$). This is one benefit of using the full flux physics based model. In addition, this approach does not require having to create special variables to deal with thermal fluid systems such as "stream varibles" in the thermal fluid Modelica models (see [17]).
3) The approximations that are convergent and dual convergent so that the resulting finite dimensional systems are suitable for optimal design and control (see [8], [9]).

In this regard it is helpful to observe that $\mathscr{A}(\varepsilon)$ is self adjoint while $\mathscr{F}(v)$ is a low order non-self adjoint operator. Consequently, one can take advantage of composite finite element - finite volume methods (see [11] and [10]) where $\mathscr{A}(\varepsilon)$ is approximated by $\mathscr{A}_N^E(\varepsilon)$ constructed by a finite element method and $\mathscr{F}(v)$ is approximated by $\mathscr{F}_N^V(v)$ constructed by a finite volume method. As noted in [11] such methods can be shown to be convergent uniformly with respect to the diffusion coefficient ε. For the controlled system, one has the additional benefit that these composite methods can be constructed to ensure dual convergence.

We have shown above that for a fixed v, as $\varepsilon \to 0^+$ the semigroup generated by $[\mathscr{A}(\varepsilon) + \mathscr{F}(v)]$ converges strongly to the semigroup generated by $\mathscr{F}(v)$. Therefore, we may think of (31) as a parabolic approximation of the limiting hyperbolic system. By employing a composite finite element - finite volume scheme, convergence is achieved even as $\varepsilon \to 0^+$. Moreover, the system is well-posed even when v approaches zero and in the limiting case one has a standard finite element approximation of the self adjoint system. The only remaining issue is how to approximate the control input operator $\mathscr{B}$. Since we are interested in small values of ε, we shall assume that the diffusion coefficient ε satisfies $0 \leq \varepsilon \leq 1$. Thus, $\mathscr{B}$ is approximated by an averaging process

$$\mathscr{B}_N = (1 - \varepsilon)\mathscr{B}_N^V + \varepsilon \mathscr{B}_N^E \tag{33}$$

where again as the notation suggests, $\mathscr{B}_N^V$ is the finite volume approximation of $\mathscr{B}$ and $\mathscr{B}_N^E$ is the finite element approximation of $\mathscr{B}$.

In the numerical results below we employ standard continuous piecewise linear finite elements for $\mathscr{A}(\varepsilon)$ and a finite volume scheme for the convective operator $\mathscr{F}(v)$. Although space does not allow us to provide complete descriptions of these approximations and their convergence properties, the interested reader can find detailed information about the finite element scheme in [18] and the finite volume (so called AVE scheme) in [14] and [19].

To construct the approximating scheme based on the idea of composite finite element - finite volume methods one creates a partition on $[0,L]$ by defining $x_j^N = jL/(N+1)$, where $j = 0,\ldots,N+1$. The standard continuous piecewise linear finite element scheme is used to construct the approximating operators $\mathscr{A}_N^E(\varepsilon)$ and $\mathscr{B}_N^E$. Since this construction appears in several books and papers we leave out the details (see [20], [21], [22], [18] and [23]). The finite volume scheme used to approximate approximation operators $\mathscr{F}_N^E(v)$ and $\mathscr{B}_N^E$ is based on the "AVE" scheme used for delay systems in [24] and [25] and may be found in [14] and [19]. To construct approximating operators using the finite volume AVE scheme for the above partition on $[0,L]$, one defines $\chi_j^N : [0,L] \to \mathbb{R}$ to be the characteristic functions on $(x_{j-1}^N, x_j^N]$, for $j = 2\ldots N+1$ and χ_1^N to be the characteristic function on $[0,x_1^N]$. Let Z_N be the closed subspace of $Z = L^2(0,L)$ defined by

$$Z_N \equiv \left\{ \varphi^N(x) = \sum_{j=1}^{N+1} z_j^N \chi_j^N(x), \quad z_j^N \in \mathbb{R} \right\}. \tag{34}$$

The orthogonal projection π_N of Z onto Z_N is defined by

$$\pi_N \varphi(\cdot) = \sum_{j=1}^{N+1} \varphi_j^N \chi_j^N(\cdot), \tag{35}$$

where for $j = 1,\ldots,N+1$,

$$\varphi_j^N \equiv \frac{N+1}{L} \int_{x_{j-1}^N}^{x_j^N} \varphi(s)ds. \tag{36}$$

are the mean values. Approximating the operator $\mathscr{F}(v)$ is now straightforward. Define $\mathscr{D}_N(v) : Z_N \to Z_N \subseteq Z$ by

$$\mathscr{D}_N(v)\varphi^N(x) = \frac{L}{N+1}\left[-v \sum_{j=1}^{N+1} [z_j^N - z_{j-1}^N]\chi_j^N(x)) \right],$$

where we set $z_0^N = 0$. Also, let $\mathscr{K}_N : Z_N \to Z_N \subseteq \mathbf{Z}$ be given by

$$\mathscr{K}_N \varphi^N(x) = -\kappa \varphi^N(x) \tag{37}$$

so that $\mathscr{F}_N^E(v) : Z_N \to Z_N \subseteq \mathbf{Z}$ is the sum

$$\mathscr{F}_N^E(v) = \mathscr{D}_N(v) + \mathscr{K}_N.$$

Recall that the input operator $\mathscr{B} : \mathbb{R}^1 \to Z$ is defined by $[\mathscr{B}u](x) = \gamma e^{-bx}u = b(x)u$. The finite volume approximating operator is simply

$$[B_N^V u](x) = [\pi_N b(x)]u.$$

Details for the counter flow heat exchanger (6) may be found in [14] where convergence and dual convergence are established.

IV. Numerical Example

Consider the LQR problem defined by the counterflow problem (6)-(9) where $L = 1$, control is applied only on the top channel with $b_1(x) = e^{-5x}u(t)$ and $b_2(x) = 0$ so that $\gamma_1 = 1$, $\gamma_2 = 0$ and $b = 5$. The diffusion coefficient is fixed at $\varepsilon = 0.005$, $\kappa_1 = 15.933$ and $\kappa_2 = 16.483$. We set $v_1 = v$ and $v_2 = (1.1)v$, where v is varied between $v = 0.1$ and $v = 0$.

The cost function is given by

$$J = \int_0^{+\infty} \left\{ \langle qQz(t,\cdot), z(t,\cdot)\rangle_Z + R[u(t)]^2 \right\} dt \tag{38}$$

where $Q = Q^* = I_Z$ is the identity operator on $Z = L^2(0,1) \times L^2(0,1)$, $R = 1 > 0$ and $q = 5$. Let $\mathscr{A}(\varepsilon,v) = [\mathscr{A}(\varepsilon) + \mathscr{F}(v) + \mathscr{K}]$. In [14] and [19] we established that the pair $(\mathscr{A}(\varepsilon,v), \mathscr{B})$ is stabilizable and there exist a unique optimal control $u^{opt}(t)$ to the LQR problem defined by the cost function (38) and

$$u^{opt}(t) = -\mathscr{K}(\varepsilon,v)z^{opt}(t).$$

Here, $\mathscr{K}(\varepsilon,v) = R^{-1}\mathscr{B}^*\Pi$ where $\Pi : Z \to Z$ is the self-adjoint bounded linear operator satisfying the standard algebraic Riccati operator equation $\mathscr{A}(\varepsilon,v)^*\Pi + \Pi\mathscr{A}(\varepsilon,v) - \Pi\mathscr{B}R^{-1}\mathscr{B}^*\Pi + Q = 0$. Moreover, the Riesz representation theorem implies that $\mathscr{K} : Z \to \mathbb{R}^1$ has the form

$$\mathscr{K}(\varepsilon,v)\begin{bmatrix} \varphi(\cdot) \\ \psi(\cdot) \end{bmatrix} = \int_0^L k_1(\varepsilon,v,\xi)\varphi(\xi)d\xi + \int_0^L k_2(\varepsilon,v,\xi)\psi(\xi)d\xi. \tag{39}$$

where $k_1(\varepsilon,v,\cdot)$ and $k_2(\varepsilon,v,\cdot)$ are called the (optimal feedback) functional gains.

Figures 2 and 3 illustrates the that the functional gains $k_1^N(\cdot)$ and $k_2^N(\cdot)$ converge as the number of finite volumes increases. Here we fixed $v = 0.1$ and refined the mesh. The plots are color codes so that $N = 128$ corresponds to the solid blue line, $N = 256$ corresponds to the solid black line, $N = 512$ corresponds to the solid red line and $N = 1{,}024$ corresponds to the dashed blue line.

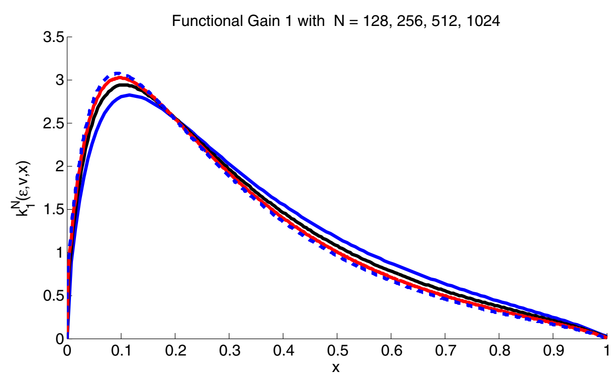

Fig. 2. Plots of $k_1(\cdot)$

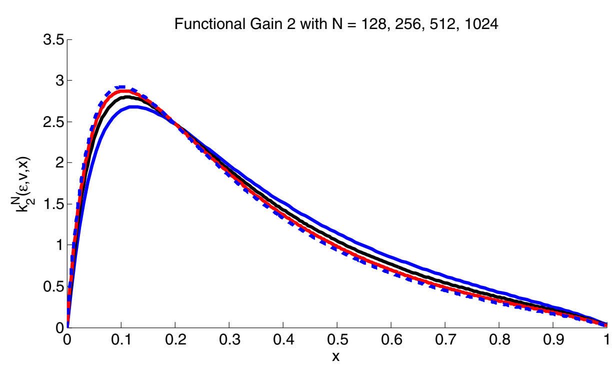

Fig. 3. Plots of $k_2(\cdot)$

Figures 4 and 5 illustrates the that the functional gains $k_1(\cdot)$ and $k_2(\cdot)$ depend continuously on v. Here we fixed $N = 64$, set $\varepsilon = 0.005$ and varied v from $v = 0.1$ down to $v = 0.0$. The plots are color codes so that $v = 0.1$ corresponds to the solid blue line, $v = 0.05$ corresponds to the solid green line, $v = 0.001$ corresponds to the solid black line and $v = 0.0$ corresponds to the solid red line. Observe that as $v \to 0^+$, the functional gains are still defined at $v = 0.0$ and both gains converge to a zero flow controller. The sharp "peak" in the $v = 0$ gains is a consequence of the finite element approximation for small ε (see [20]). Finally, the functional gains also approach a limiting controller for a fixed $\hat{v} > 0$, as $\varepsilon \to 0^+$. This is a consequence of the results presented in [19].

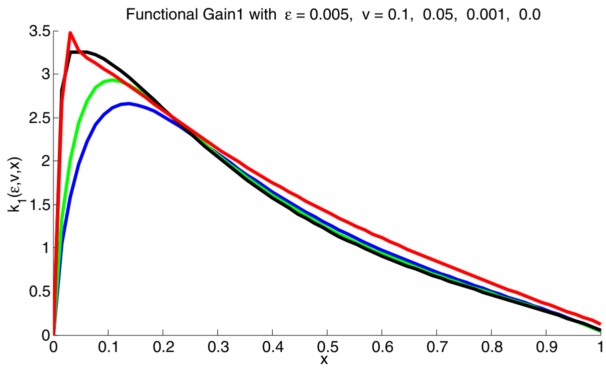

Fig. 4. Plots of $k_1(\cdot)$

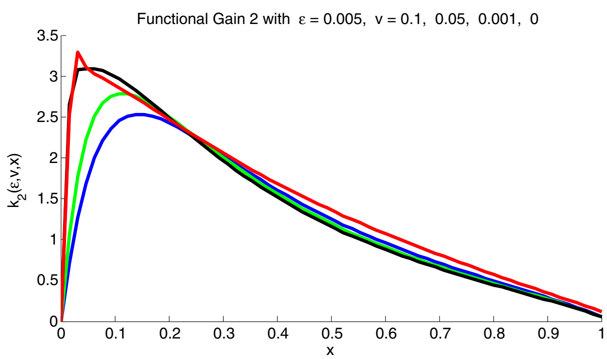

Fig. 5. Plots of $k_2(\cdot)$

V. CONCLUSIONS

We used the so called full flux model to approximate the physics of heat exchangers. This approach allows for the development of approximate finite dimensional models that are valid for low or zero flow. Moreover the composite finite element-finite volume method preserves important systems properties needed for simulation and optimal control design for a range of Péclet numbers. A simple numerical example was provided to illustrate convergence of the optimal feedback gain operators as $v \to 0^+$. A complete analysis of the method and several additional numerical examples will appear in a forthcoming paper.

VI. ACKNOWLEDGMENTS

This research was supported in part by the DOE contract DE-EE0004261 under subcontract # 4345-VT-DOE-4261 from Penn State University.

REFERENCES

[1] A. Maidi, M. Diaf, and J.-P. Corriou, "Boundary geometric control of a counter-current heat exchanger," *Journal of Process Control*, vol. 19, no. 2, pp. 297–313, 2009.

[2] ——, "Boundary control of a parallel-flow heat exchanger by input–output linearization," *Journal of Process Control*, vol. 20, no. 10, pp. 1161–1174, 2010.

[3] A. Michel and A. Kugi, "Accurate low-order dynamic model of a compact plate heat exchanger," *International Journal of Heat and Mass Transfer*, vol. 61, pp. 323–331, 2013.

[4] D. Taler, M. Trojan, and J. Taler, "Mathematical modelling of tube heat exchangers with complex flow arrangement," *Chemical and Process Engineering*, pp. 7–19, 2011.

[5] H. Sano, "Exponential stability of a mono-tubular heat exchanger equation with output feedback," *Systems & control letters*, vol. 50, no. 5, pp. 363–369, 2003.

[6] ——, "Observability and reachability for parallel-flow heat exchanger equations," *IMA Journal of Mathematical Control and Information*, vol. 24, pp. 137–147, 2007.

[7] ——, "On reachability of parallel-flow heat exchanger equations with boundary inputs," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 83, no. 1, pp. 1–4, 2007.

[8] J. A. Burns, K. Ito, and G. Propst, "On nonconvergence of adjoint semigroups for control systems with delays," *SIAM J. Control and Optimization*, vol. 26, pp. 1442–1454, 1988.

[9] J. A. Burns, "Nonlinear distributed parameter control systems with non-normal linearizations: Applications and approximations," *Frontiers in Applied Mathematics*, vol. 27, pp. 17–53, 2003.

[10] P. Deuring, R. Eymard, and M. Mildner, "L2-stability independent of diffusion for a finite element–finite volume discretization of a linear convection-diffusion equation," *Preprint*, 2014.

[11] P. Deuring and M. Mildner, "Stability of a combined finite element-finite volume discretization of convection-diffusion equations," *Numerical Methods for Partial Differential Equations*, vol. 28, no. 2, pp. 402–424, 2012.

[12] H. Sano and S. Nakagiri, "Feedback control of parallel-flow heat exchanger equations using a forwardstepping method," in *Proceedings of the 14th WSEAS International Conference on Applied Mathematics*. World Scientific and Engineering Academy and Society (WSEAS), 2009, pp. 106–111.

[13] T. Besson, A. Tchousso, and C. Xu, "Exponential stability of a class of hyperbolic pde models from chemical engineering," in *45th IEEE Conference on Decision and Control*, 2006, pp. 3974–3978.

[14] J. A. Burns and E. M. Cliff, "Numerical methods for optimal control of heat exchangers," in *2014 American Control Conference*, 2014, pp. 1649–1654.

[15] R. Curtain and H. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer, 1995.

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.

[17] F. Casella and R. Franke, "Modelica.fluid.usersguide," 2009, *Available at http://www.maplesoft.com/documentation_center/online_manuals/modelica/Modelica_Fluid_UsersGuide.html*.

[18] G. Strang and G. J. Fix, *An analysis of the finite element method*. Prentice-Hall Englewood Cliffs, 1973, vol. 212.

[19] J. A. Burns and E. M. Cliff, "Control of hyperbolic pde systems with actuator dynamics," in *IEEE 53rd Conference on Decision and Control*, 2014, pp. 2864–2869.

[20] J. Burns and L. Zietsman, "Upwind approximations and mesh independence for LQR control of convection diffusion equations," in *46th IEEE Conference on Decision and Control*, 2007, pp. 219–224.

[21] J. Burns, E. Sachs, and L. Zietsman, "Mesh Independence of Kleinman–Newton Iterations for Riccati Equations in Hilbert Space," *SIAM Journal on Control and Optimization*, vol. 47, pp. 2663–2692, 2008.

[22] C. A. Fletcher, *Computational galerkin methods*. Springer, 1984.

[23] E. D. Vugrin, "On approximation and optimal control of nonnormal distributed parameter systems," Ph.D. dissertation, Virginia Polytechnic Institute and State University, 2004, *Available at http://scholar.lib.vt.edu/theses/available/etd-04282004-164740/unrestricted/dissert.pdf*.

[24] H. T. Banks and J. A. Burns, "Hereditary control problems: Numerical methods based on averaging approximations," *SIAM J. Control Optim*, vol. 16, no. 2, pp. 169–208, 1978.

[25] H. T. Banks, J. A. Burns, and E. M. Cliff, "Parameter estimation and identification for systems with delays," *SIAM Journal on Control and Optimization*, vol. 19, p. 791, 1981.